Partial Fraction Expansion Examples

Consider the causal system function

$$H(z) = \frac{2 + 3z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}$$

- Find the impulse response $h[n]$

- This system has two real poles $p_1 = 1/2$ and $p_2 = -1/4$
- Since $H(z)$ is proper rational we can expand as

$$H(z) = \frac{2 + 3z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 + \frac{1}{4}z^{-1}}$$

- Solving for the coefficients:

$$A_1 = \frac{2 + 3z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}\left(1 - \frac{1}{2}z^{-1}\right)\Bigg|_{z=1/2} = \frac{2 + 3z^{-1}}{1 + \frac{1}{4}z^{-1}}\Bigg|_{z^{-1}=2} = \frac{2+6}{1+\frac{3}{4}} = \frac{8}{6/4} = \frac{16}{3}$$

$$A_2 = \frac{2 + 3z^{-1}}{1 - \frac{1}{2}z^{-1}}\Bigg|_{z^{-1}=-4} = \frac{2-12}{1+2} = \frac{-10}{3}$$

so we have

$$H(z) = \frac{16/3}{1 - \frac{1}{2}z^{-1}} + \frac{-10/3}{1 + \frac{1}{4}z^{-1}}$$

and

$$h[n] = \frac{16}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{10}{3}\left(-\frac{1}{4}\right)^n u[n]$$

- Check in MATLAB

```
>> [A,p,K] = residuez([2 3],conv([1 -1/2],[1 1/4]))

A = 5.3333  % = 16/3 as expected
   -3.3333  % = -10/3 as expected
p = 0.5000
   -0.2500
K = []
```

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Consider the causal system function

$$H(z) = \frac{2+3z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

- For $x[n] = (1/3)^n u[n]$ find the output $y[n]$

- To find $y[n]$ we work in the z-domain using $Y(z) = X(z)H(z)$ and then inverse z-transform
- The z-transform of $x[n]$ is

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

- Since $Y(z)$ is proper rational we can expand as

$$Y(z) = \frac{2+3z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{A_1}{1-\frac{1}{3}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}} + \frac{A_3}{1+\frac{1}{4}z^{-1}}$$

- Solving for the coefficients:

$$A_1 = \left.\frac{2+3z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=3} = \frac{2+9}{\left(1-\frac{3}{2}\right)\left(1+\frac{3}{4}\right)} = \frac{11}{\frac{-1}{2}\cdot\frac{7}{4}} = \frac{-88}{7}$$

$$A_2 = \left.\frac{2+3z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=2} = \frac{2+6}{\left(1-\frac{2}{3}\right)\left(1+\frac{1}{2}\right)} = \frac{8}{\frac{-1}{\not{3}}\cdot\frac{\not{3}}{2}} = 16$$

$$A_3 = \left.\frac{2+3z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=-4} = \frac{2-12}{\left(1+\frac{4}{3}\right)\left(1+\frac{4}{2}\right)} = \frac{-10}{\frac{7}{\not{3}}\cdot\not{3}} = \frac{-10}{7}$$

so we have

$$Y(z) = \frac{7}{1-\frac{1}{3}z^{-1}} + \frac{16}{1-\frac{1}{2}z^{-1}} + \frac{-10/7}{1+\frac{1}{4}z^{-1}}$$

and

$$y[n] = \frac{-88}{7}\left(\frac{1}{3}\right)^n u[n] + 16\left(\frac{1}{2}\right)^n u[n] - \frac{10}{7}\left(-\frac{1}{4}\right)^n u[n]$$

- We can check this using the MATLAB `[A,p,K] = residuez(b,a)` function

```
[A,p,K] = residuez([2 3],conv([1 -1/3],conv([1 -1/2],[1 1/4])))
```