Sampling and Aliasing

Chapter 4

With this chapter we move the focus from signal modeling and analysis, to converting signals back and forth between the analog (continuous-time) and digital (discrete-time) domains. Back in Chapter 2 the systems blocks C-to-D and D-to-C were introduced for this purpose. The question is, how must we choose the sampling rate in the C-to-D and D-to-C boxes so that the analog signal can be reconstructed from its samples.

The lowpass sampling theorem states that we must sample at a rate, f_s, at least twice that of the highest frequency of interest in analog signal $x(t)$. Specifically, for $x(t)$ having spectral content extending up to B Hz, we choose $f_s = 1/T_s > 2B$ in forming the sequence of samples

$$x[n] = x(nT_s), -\infty < n < \infty. \tag{4.1}$$

Sampling

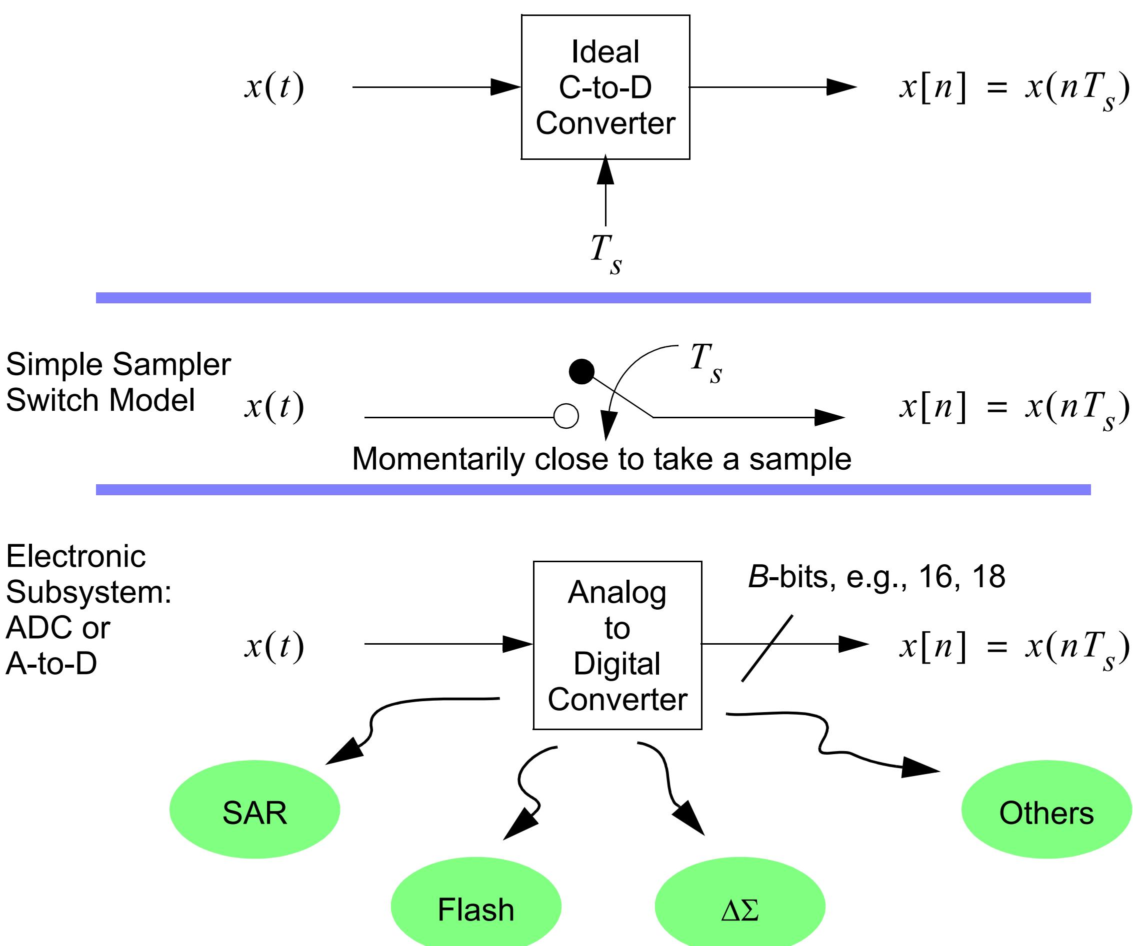
- We have spent considerable time thus far, with the continuous-time sinusoidal signal

$$x(t) = A\cos(\omega t + \phi), \tag{4.2}$$

  where t is a continuous variable

- To manipulate such signals in MATLAB or any other computer too, we must actually deal with samples of $x(t)$

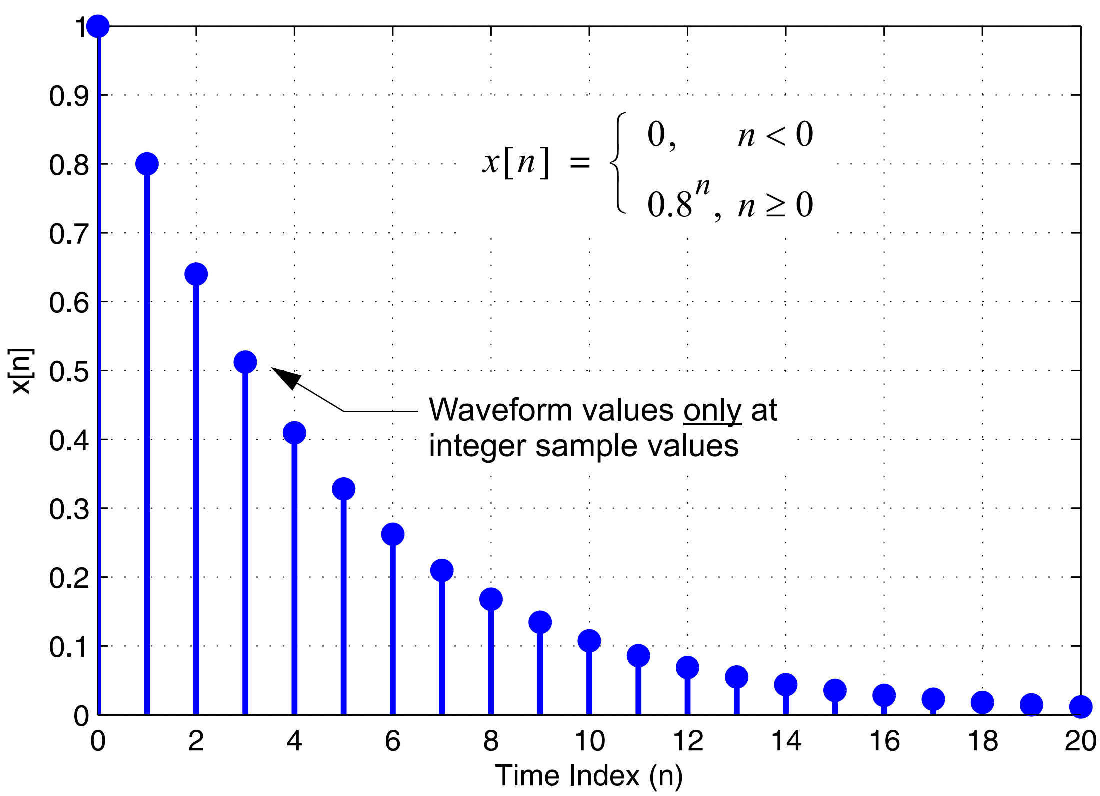
- Recall from the course introduction, that a discrete-time signal can be obtained by uniformly sampling a continuous-time signal at $t_n = nT_s$, i.e., $x[n] = x(nT_s)$
  - The values $x[n]$ are *samples* of $x(t)$
  - The time interval between samples is T_s
  - The *sampling rate* is $f_s = 1/T_s$
  - Note, we could write $x[n] = x(n/f_s)$
- A system which performs the sampling operation is called a *continuous-to-discrete* (C-to-D) converter



SAR = successive approximation register
$\Delta\Sigma$ = delta-sigma modulator (oversampling)

- A real C-to-D has imperfections, with careful design they can be minimized, or at least have negligible impact on overall system performance
- For testing and simulation only environments we can easily generate discrete-time signals on the computer, with no need to actually capture and C-to-D process a live analog signal
- In this course we depict discrete-time signals as a sequence, and plot the corresponding waveform using MATLAB's stem function, sometimes referred to as a *lollypop* plot

```
>> n = 0:20;
>> x = 0.8.^n;
>> stem(n,x,'filled','b','LineWidth',2)
>> grid
>> xlabel('Time Index (n)')
>> ylabel('x[n]')
```



Sampling Sinusoidal Signals

- We will continue to find sinusoidal signals to be useful when operating in the discrete-time domain
- When we sample (4.2) we obtain a sinusoidal sequence

$$\begin{aligned} x[n] &= x(nT_s) \\ &= A\cos(\omega n T_s + \phi) \\ &= A\cos(\hat{\omega} n + \phi) \end{aligned} \tag{4.3}$$

- Notice that we have defined a new frequency variable

$$\hat{\omega} \equiv \omega T_s = \frac{\omega}{f_s} \text{ rad}, \tag{4.4}$$

known as the *discrete-time frequency* or normalized continuous-time frequency

  – Note that $\hat{\omega}$ has units of radians, but could also be called radians/sample, to emphasize the fact that sampling is involved

  – Note also that many values of ω map to the same $\hat{\omega}$ value by virtue of the fact that T_s is a system parameter that is not unique either

  – Since $\omega = 2\pi f$, we could also define $\hat{f} \equiv fT_s$ as the discrete-time frequency in cycles/sample

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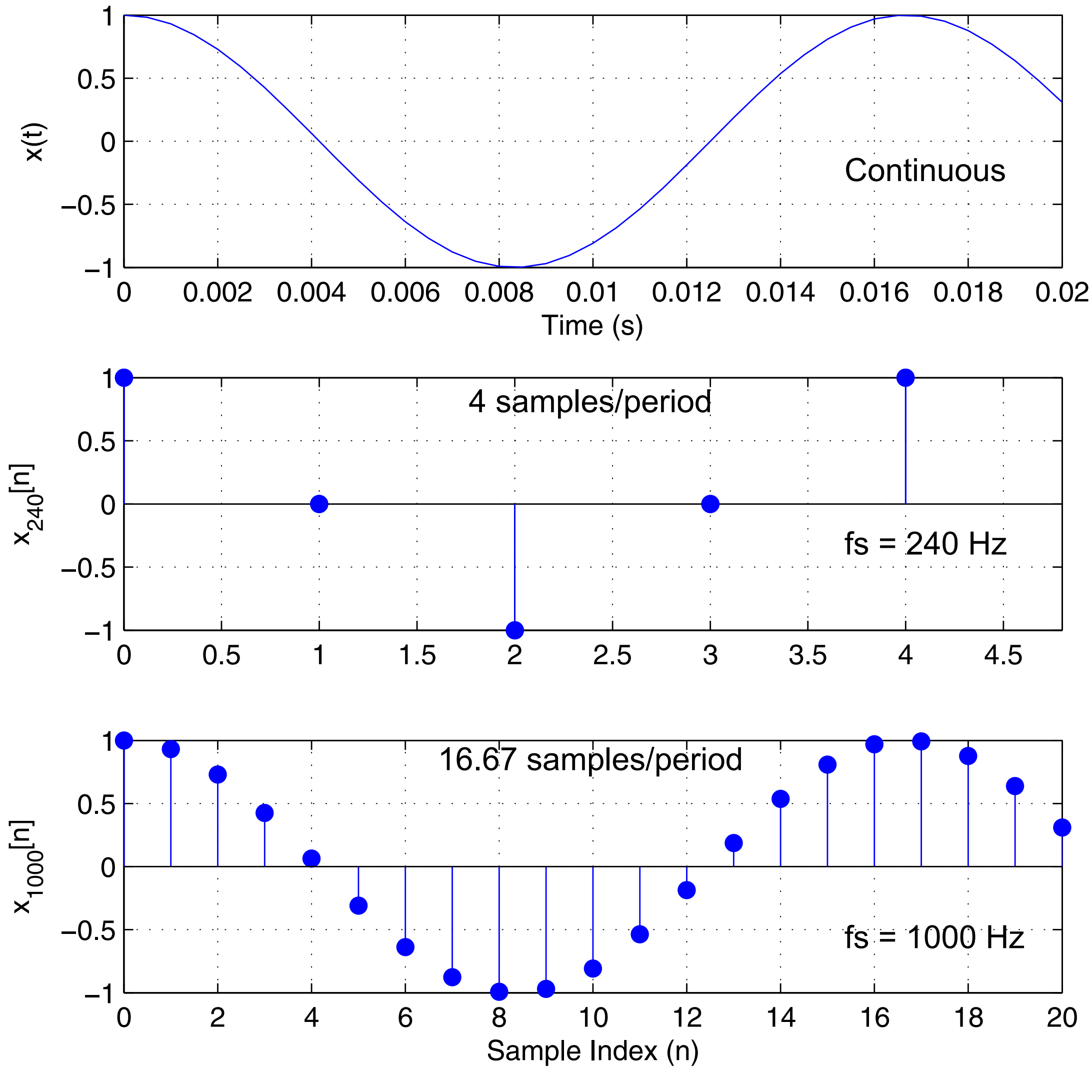
Example: Sampling Rate Comparisons

- Consider $x(t) = \cos(2\pi \cdot 60 \cdot t)$ at sampling rates of 240 and 1000 samples per second

– The corresponding sample spacing values are

$$T_s = \frac{1}{240} = 4.1666 \text{ ms} \qquad T_s = \frac{1}{1000} = 1 \text{ ms}$$

```
>> t = 0:1/2000:.02;
>> x = cos(2*pi*60*t); % approx. to continuous-time
>> t240 = 0:1/240:.02;
>> n240 = 0:length(t240)-1;
>> x240 = cos(2*pi*60/240*n240); % fs = 240 Hz
>> axis([0 4.8 -1 1]) % axis scale since .02*240 = 4.8
>> t1000 = 0:1/1000:.02;
>> n1000 = 0:length(t1000)-1;
>> x1000 = cos(2*pi*60/1000*n1000); % fs = 1000 Hz
```



- The analog frequency is $2\pi \cdot 60$ rad/s or 60 Hz
- When sampling with $f_s = 240$ and 1000 Hz

$$\hat{\omega}_{240} = 2\pi \cdot 60/240 = 2\pi(0.25) \text{ rad}$$

$$\hat{\omega}_{1000} = 2\pi \cdot 60/1000 = 2\pi(0.06) \text{ rad}$$

  respectively
- The sinusoidal sequences are

$$x_{240}[n] = \cos(0.5\pi n)$$

$$x_{1000}[n] = \cos(0.12\pi n)$$

  respectively
- It turns out that we can reconstruct the original $x(t)$ from either sequence
- Are there other continuous-time sinusoids that when sampled, result in the same sequence values as x_{240} and x_{1000}?
- Are there other sinusoid sequences of different frequency $\hat{\omega}$ that result in the same sequence values?

The Concept of Aliasing

- In this section we begin a discussion of the very important signal processing topic known as *aliasing*
- **Alias** as found in the Oxford American dictionary: *noun*
  - A false or assumed identity: a spy operating under an alias.
  - Computing: an alternative name or label that refers to a file, command, address, or other item, and can be used to locate or access it.

- <u>Telecommunications</u>: each of a set of signal frequencies that, when sampled at a given uniform rate, would give the same set of sampled values, and thus might be incorrectly substituted for one another when reconstructing the original signal.

- Consider the sinusoidal sequence

$$x_1[n] = \cos(0.4\pi n) \tag{4.5}$$

  - Clearly, $\hat{\omega} = 0.4\pi$

- We know that cosine is a mod 2π function, so

$$\begin{aligned} x_2[n] &= \cos(2.4\pi n) \\ &= \cos[(2+0.4)\pi n] = \cos(0.4\pi n + 2\pi n) \\ &= \cos(0.4\pi n) = x_1[n] \end{aligned} \tag{4.6}$$

  - We see that $\hat{\omega} = 2.4\pi$ gives the same sequence values as $\hat{\omega} = 0.4\pi$, so 2.4π and 0.4π are aliases of each other

- We can generalize the above to any 2π multiple, i.e.,

$$\hat{\omega}_l = \ddot{\omega}_0 + 2\pi l,\ l = 0, 1, 2, 3, \ldots \tag{4.7}$$

  result in identical frequency samples for $\cos(\hat{\omega}_l n)$ due to the mod 2π property of sine and cosine

- We can take this one step further by noting that since $\cos(\theta) = \cos(-\theta)$, we can write

$$\begin{aligned} x_3[n] &= \cos(1.6\pi n) \\ &= \cos[(2-0.4)\pi n] = \cos(2\pi n - 0.4\pi n) \\ &= \cos(-0.4\pi n) = \cos(0.4\pi n) \end{aligned} \tag{4.8}$$

– We see that $\hat{\omega} = 1.6\pi$ gives the same sequence values as $\hat{\omega} = 0.4\pi$, so 1.6π and 0.4π are aliases of each other

- We can generalize this result to saying

$$\hat{\omega}_l = 2\pi l - \hat{\omega}_0, l = 0, 1, 2, 3, \ldots \tag{4.9}$$

result in identical frequency samples for $\cos(\hat{\omega}_l n)$ due to the mod 2π property and evenness property of cosine

– This result also holds for sine, expect the amplitude is inverted since $\sin(-\theta) = -\sin(\theta)$

- In summary, for any integer *l*, and discrete-time frequency $\hat{\omega}_0$, the frequencies

$$\hat{\omega}_0, \hat{\omega}_0 + 2\pi l, 2\pi l - \hat{\omega}_0, l = 1, 2, 3, \ldots \tag{4.10}$$

all produce the same sequence values with cosine, and with sine may differ by the numeric sign

– A generalization to handle both cosine and sine is to consider the inclusion of an arbitrary phase ϕ,

$$\begin{aligned} A\cos[(\hat{\omega} + 2\pi l)n + \phi] &= A\cos[\hat{\omega}n + 2\pi l \cdot n + \phi] \\ &= A\cos(\hat{\omega}n + \phi) \\ A\cos[(2\pi l - \hat{\omega})n - \phi] &= A\cos[2\pi l \cdot n - \hat{\omega}n - \phi] \\ &= A\cos(-\hat{\omega}n - \phi) \\ &= A\cos(\hat{\omega}n + \phi) \end{aligned} \tag{4.11}$$

– Note in the second grouping the sign change in the phase

- The frequencies of (4.10) are *aliases* of each other, in terms of discrete-time frequencies

- The smallest value, $\hat{\omega} \in [0, \pi)$, is called the *principal alias*
- These aliased frequencies extend to sampling a continuous-time sinusoid using the fact that $\hat{\omega} = \omega T_s$ or $\omega = \hat{\omega}/T_s$ $= \hat{\omega} f_s$, thus we may rewrite (4.10) in terms of the continuous-time frequency ω_0

$$\omega_0, \omega_0 + 2\pi l f_s, 2\pi l f_s - \omega_0, l = 1, 2, 3 \ldots \tag{4.12}$$

  – In terms of frequency in Hz we also have

$$f_0, f_0 + l f_s, l f_s - f_0, l = 1, 2, 3 \ldots \tag{4.13}$$

- When viewed in the continuous-time domain, this means that sampling $A\cos(2\pi f_0 t + \phi)$ with $t \rightarrow nT_s$ results in

$$\begin{aligned} A\cos[2\pi f_0(nT_s) + \phi] &= A\cos[2\pi(f_0 + l f_s)(nT_s) + \phi] \\ &= A\cos[2\pi(l f_s - f_0)(nT_s) - \phi] \end{aligned} \tag{4.14}$$

  being equivalent sequences for any n and any l

Example: Input a 60 Hz, 340 Hz, or 460 Hz Sinusoid with $f_s = 400$ Hz

- The analog signal is

$$x_1(t) = \cos(2\pi 60 t + \pi/3)$$

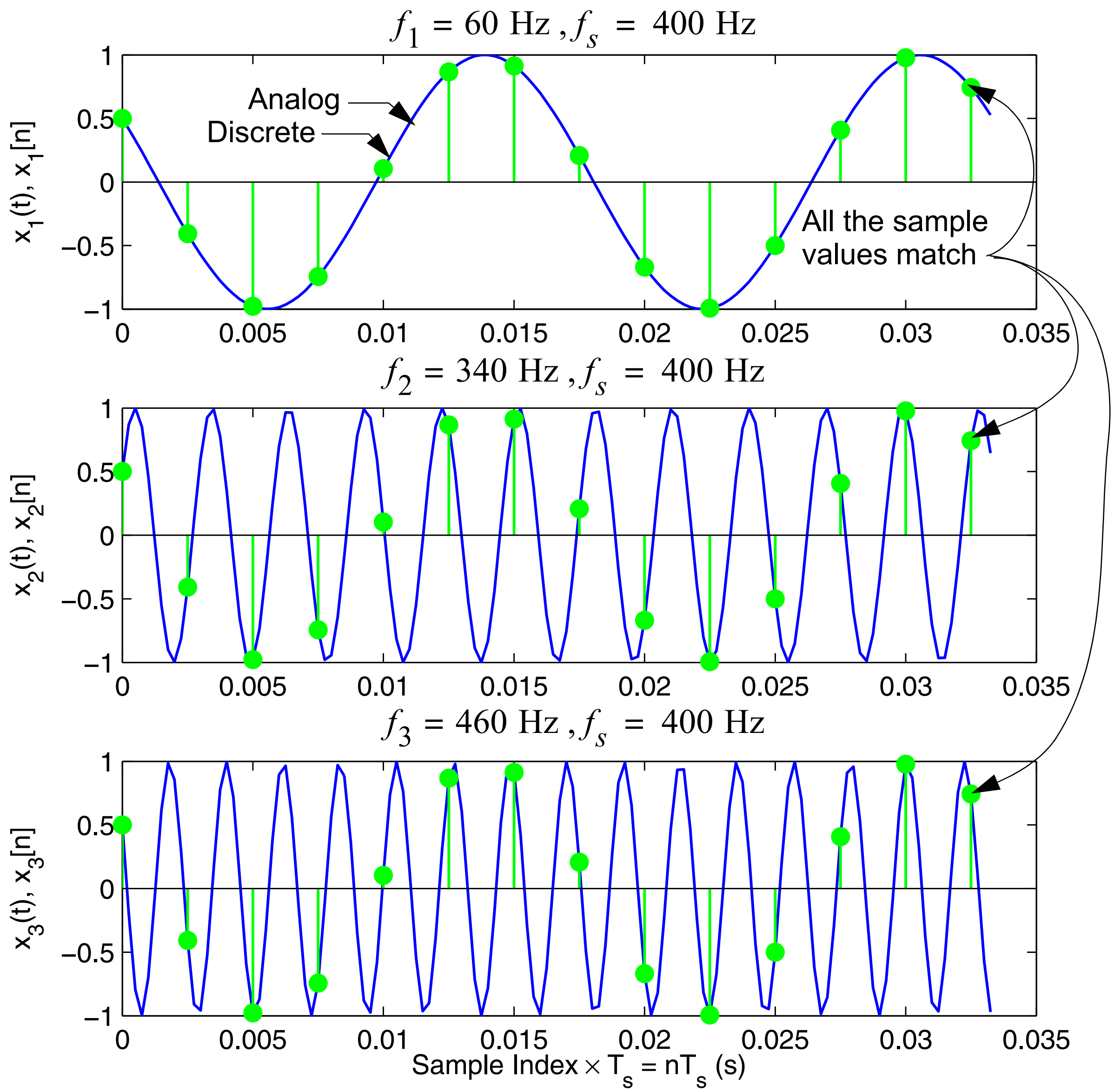
$$x_2(t) = \cos(2\pi 340 t - \pi/3)$$

$$x_3(t) = \cos(2\pi 460 t + \pi/3)$$

- We sample $x_i(t)$, $i = 1, 2, 3$ at rate $f_s = 400$ Hz

```
>> ta = 0:1/4000:2/60; % analog time axis
>> xa1 = cos(2*pi*60*ta+pi/3);
>> xa2 = cos(2*pi*340*ta-pi/3);
```

```
>> xa3 = cos(2*pi*460*ta+pi/3);
>> tn = 0:1/400:2/60; % discrete-time axis as n*Ts
>> xn1 = cos(2*pi*60*tn+pi/3);
>> xn2 = cos(2*pi*340*tn-pi/3);
>> xn3 = cos(2*pi*460*tn+pi/3);
```



- We have used (4.14) to set this example up, so we expected the sample values for all three signals to be identical
- This shows that 60, 340, and 460, are aliased frequencies, when the sampling rate is 400 Hz

– Note: 400 - 340 = 60 Hz and 460 - 400 = 60 Hz

- The discrete-time frequencies are $\omega_i = 0.3\pi, 1.7\pi, 2.3\pi$
  – Note: $2\pi - 1.7\pi = 0.3\pi$ rad and $2.3\pi - 2\pi = 0.3\pi$ rad

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Example: $5\cos(7.3\pi n + \pi/4)$ versus $5\cos(0.7\pi n + \pi/4)$

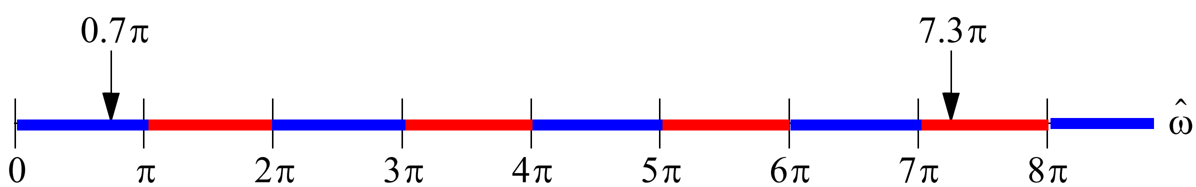
- To start with we need to see if either

$$7.3\pi = 0.7\pi + 2\pi l$$

$$\text{or } 7.3\pi = 2\pi l - 0.7\pi$$

  for l a positive integer

- Solving the first equation, we see that $l = 3.3$, which is not an integer
- Solving the second equation, we see that $l = 4$, which is an integer

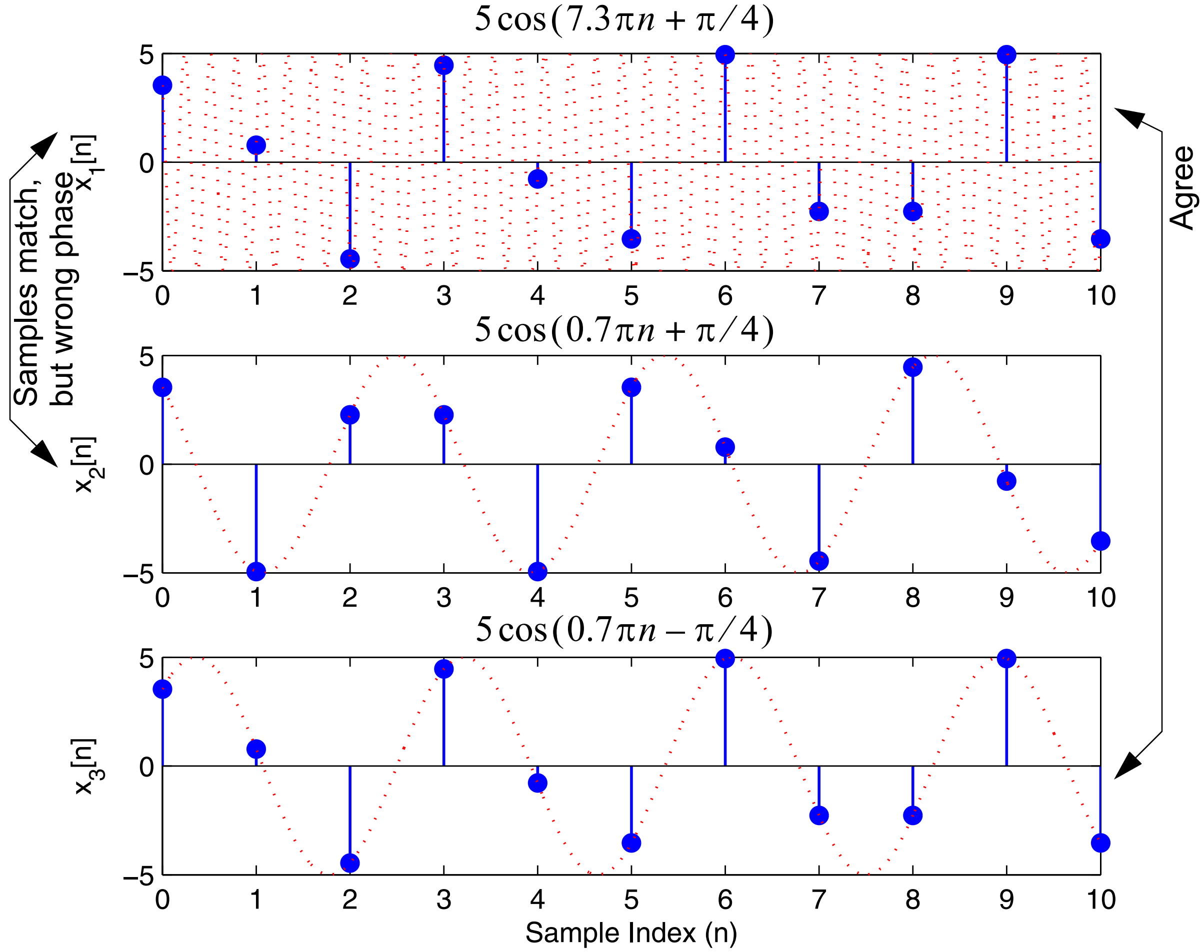


What are some other valid alias frequencies?

- The phase does not agree with (4.11), so we will use MATLAB to see if $5\cos(0.7\pi n + \pi/4) \to 5\cos(0.7\pi n - \pi/4)$ to make the samples agree in a time alignment sense

```
>> n = 0:10; % discrete time axis
>> x1 = 5*cos(7.3*pi*n+pi/4);
>> x2 = 5*cos(0.7*pi*n+pi/4);
>> x3 = 5*cos(0.7*pi*n-pi/4);
>> na = 0:1/200:10; % continuous time axis
>> x1a = 5*cos(7.3*pi*na+pi/4);
```

```
>> x2a = 5*cos(0.7*pi*na+pi/4);
>> x3a = 5*cos(0.7*pi*na-pi/4);
```



The Spectrum of a Discrete-Time Signal

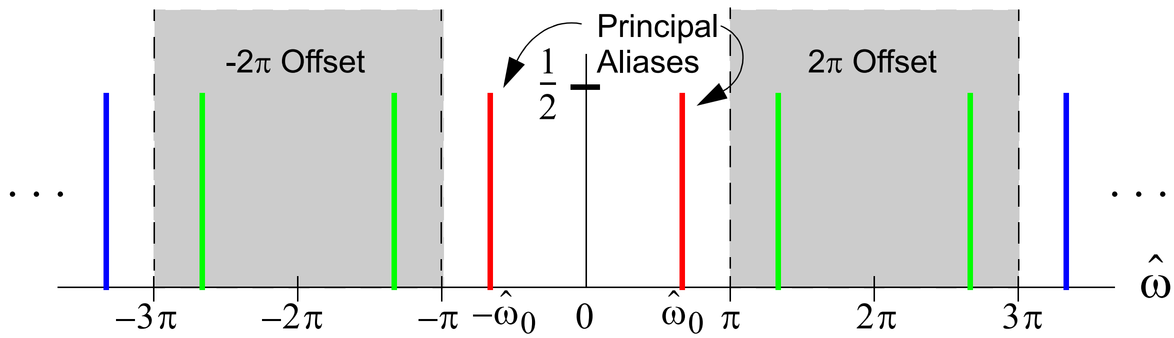
- As alluded to in the previous example, a spectrum plot can be helpful in understanding aliasing
- From the earlier discussion of line spectra, we know that for each real cosine at $\hat{\omega}_0$, the result is spectral lines at $\pm\hat{\omega}_0$
- When we consider the aliased frequency possibilities for a single real cosine signal, we have spectral lines not only at $\pm\hat{\omega}_0$, but at all $\pm 2\pi l$ frequency offsets, that is

$$\pm\hat{\omega}_0 \pm 2\pi l, l = 0, 1, 2, 3, \ldots \tag{4.15}$$

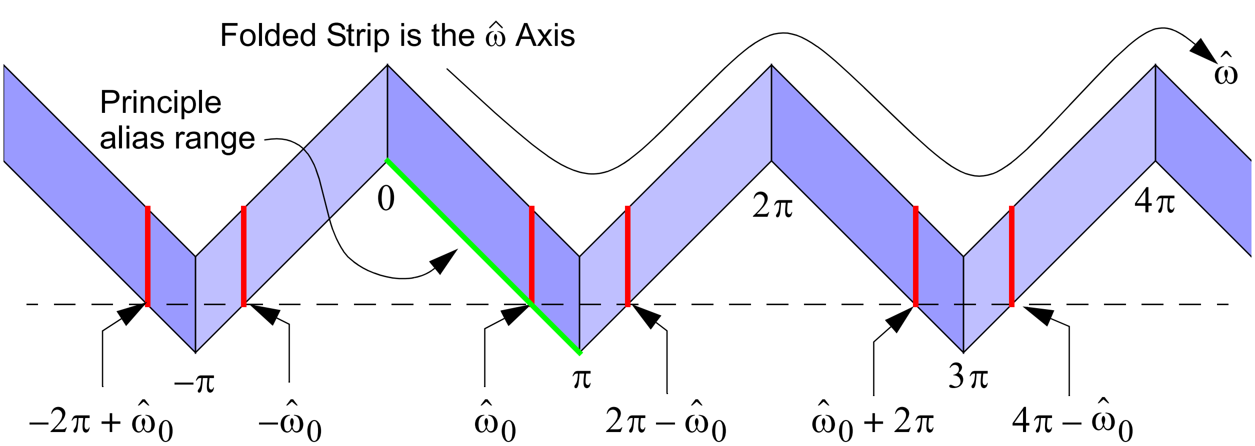
- The principal aliases occur when $l = 0$, as these are the only frequencies on the interval $[-\pi, \pi)$

<u>Example</u>: $x[n] = \cos(0.4\pi n)$

- The line spectra plot of this discrete-time sinusoid is shown below



- A particularly useful view of the alias frequencies is to consider a folded strip of paper, with folds at integer multiples of π, with the strip representing frequencies along the ω-axis



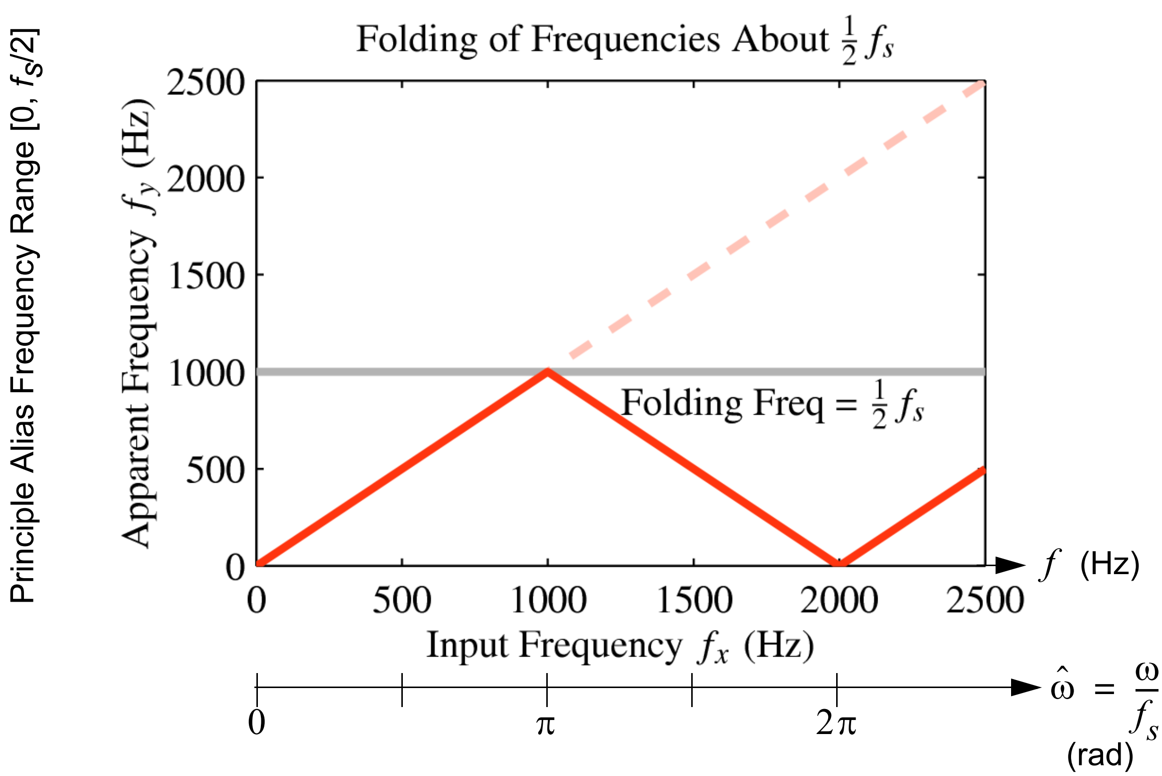
All of the alias frequencies are on a the same line when the paper is folded like an accordian, hence the term *folded frequencies.*

The Sampling Theorem

- The lowpass sampling theorem states that we must sample at a rate, f_s, at least twice that of the highest frequency in the analog signal $x(t)$. Specifically, for $x(t)$ having spectral content extending up to B Hz, we must choose $f_s = 1/T_s > 2B$

Example: Sampling with $f_s = 2000\,\text{Hz}$

- When we sample an analog signal at 2000 Hz the maximum usable frequency range (positive frequencies) is 0 to $f_s/2\,\text{Hz}$
- This is a result of the sampling theorem, which says that we must sample at a rate that is twice the highest frequency to avoid aliasing; in this case 1000 Hz is that maximum
- If the signal being sampled violates the sampling theorem, aliasing will occur (see the figure below)



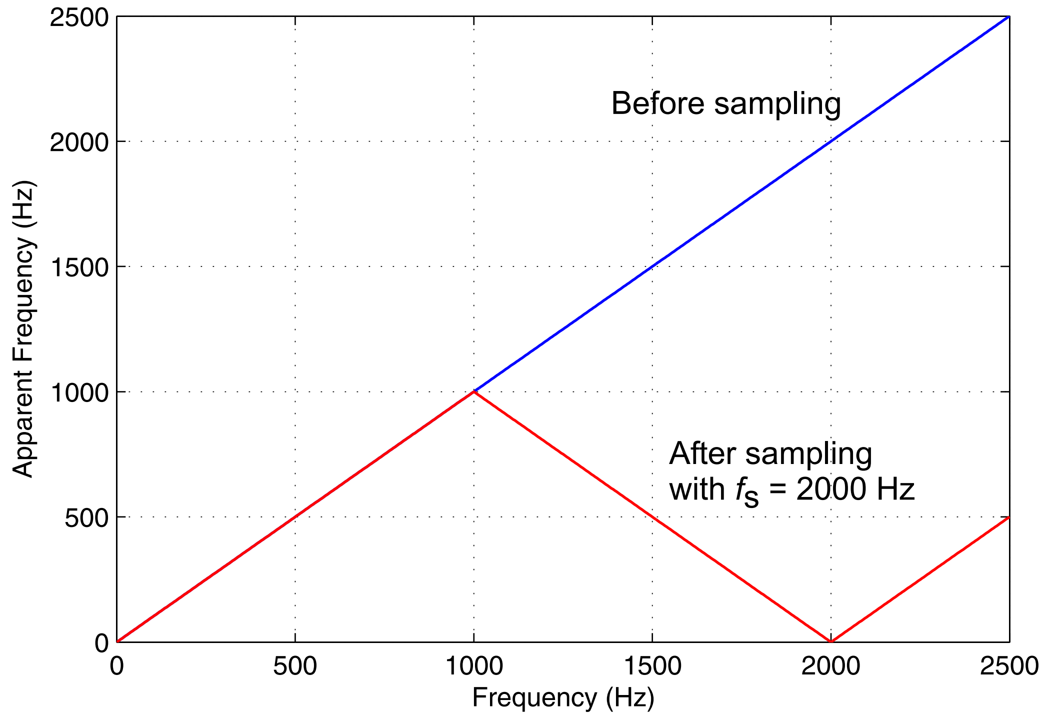
- An input frequency of 1500 Hz aliases to 500 Hz, as does an input frequency of 2500 Hz
- The behavior of input frequencies being converted to principle value alias frequencies, continues as f increases
- Notice also that the discrete-time frequency axis can be displayed just below the continuous-time frequency axis, using the fact that $\hat{\omega} = 2\pi f / f_s$ rad
- We can just as easily map from the ω–axis back to the continuous-time frequency axis via $f = \hat{\omega} f_s / (2\pi)$
- Working this in MATLAB we start by writing a support function

```
function f_out = prin_alias(f_in,fs)
% f_out = prin_alias(f_in,fs)
%
% Mark Wickert, October 2006

f_out = f_in;

for n=1:length(f_in)
    while f_out(n) > fs/2
        f_out(n) = abs(f_out(n) - fs);
    end
end
```

- We now create a frquency vector that sweeps from 0 to 2500 and assume that $f_s = 2000$ Hz

```
>> f = 0:5:2500;
>> f_alias = prin_alias(f,2000);
>> plot(f,f)
>> hold
```

```
Current plot held
>> plot(f,f_alias,'r')
>> grid
>> xlabel('Frequency (Hz)')
>> ylabel('Apparent Frequency (Hz)')
>> print -tiff -depsc f_alias.eps
```



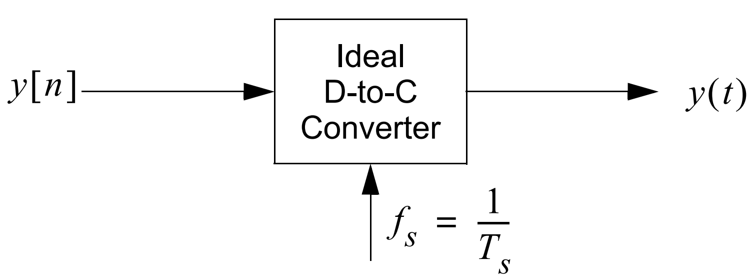
Example: Compact Disk Digital Audio

- Compact disk (CD) digital audio uses a sampling rate of $f_s = 44.1$ kHz
- From the sampling theorem, this means that signals having frequency content up to 22.05 kHz can be represented
- High quality audio signal processing equipment generally has an upper frequency limit of 20 kHz
  - Musical instruments can easily produce harmonics above 20 kHz, but human's cannot hear these signals

- The fact that aliasing occurs when the sampling theorem is violated leads us to the topic of reconstructing a signal from its samples
- In the previous example with $f_s = 2000\,\text{Hz}$, we see that taking into account the principle alias frequency range, the usable frequency band is only [0, 1000] Hz

Ideal Reconstruction

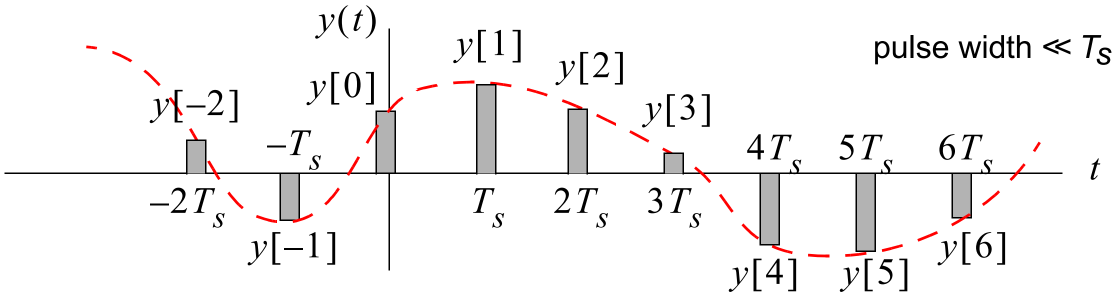
- Reconstruction refers using just the samples $x[n] = x(nT_s)$ to return to the original continuous-time signal $x(t)$
- *Ideal* reconstruction refers to exact reconstruction of $x(t)$ from its samples so long as the sampling theorem is satisfied
- In the extreme case example, this means that a sinusoid having frequency just less than $f_s/2$, can be reconstructed from samples taken at rate f_s
- The block diagram of an ideal discrete-to-continuous (D-to-C) converter is shown below



- In very simple terms the D-to-C performs interpolation on the sample values $y[n]$ as they are placed on the time axis at spacing T_s s

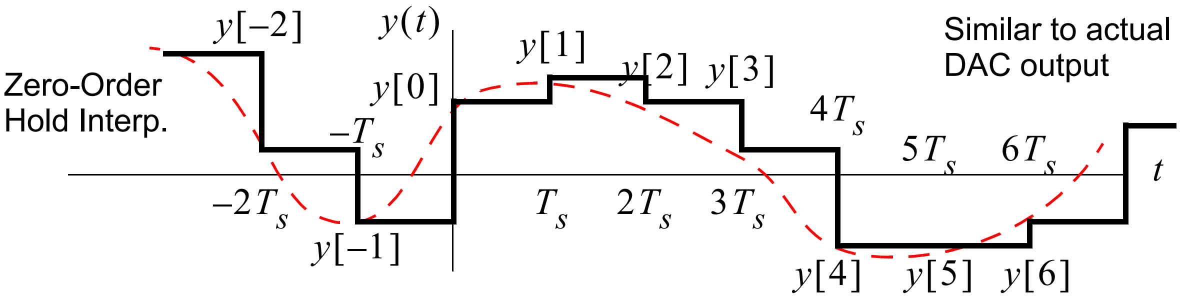
– There is an ideal interpolation function that is discussed in detail in Chapter 12 of the text

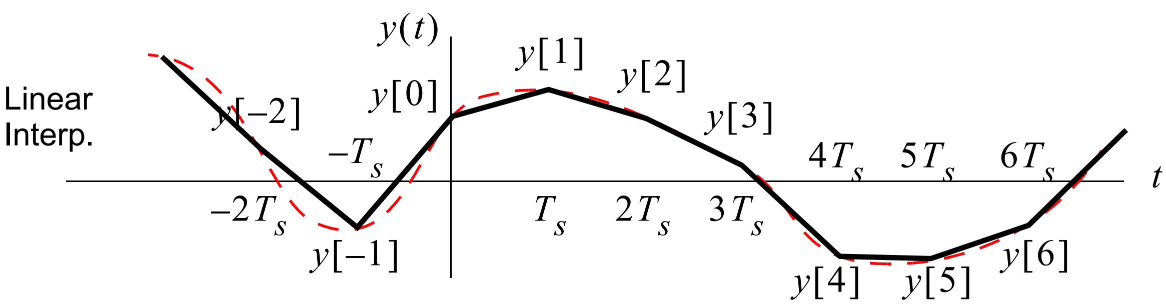
- Consider placing the sample values directly on the time axis



- The D-to-C places the $y[n]$ values on the time axis and then must interpolate signal waveform values in between the sequence (sample) values
- Two very simple interpolation functions are *zero-order hold* and *linear interpolation*
- With zero-order hold each sample value is represented as a rectangular pulse of width T_s and height $y[n]$
  – Real world digital-to-analog converters (DACs) perform this type of interpolation
- With linear interpolation the continuous waveform values between each sample value are formed by connecting a line

between the $y[n]$ values



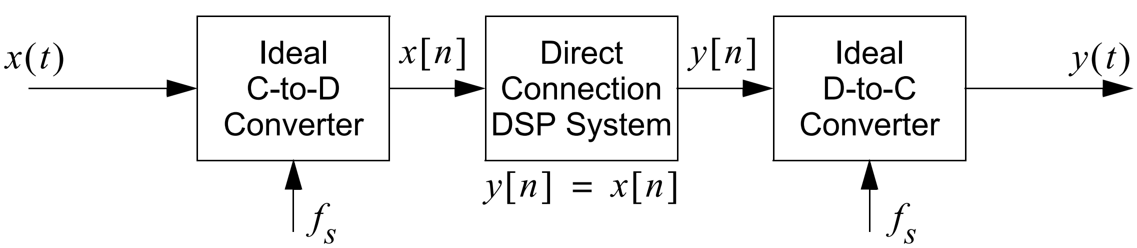


- Both cases introduce errors, so it is clear that something better must exist
- For D-to-C conversion using pulses, we can write

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s) \tag{4.16}$$

where $p(t)$ is a rectangular pulse of duration T_s

- A complete sampling and reconstruction system requires both a C-to-D and a D-to-C

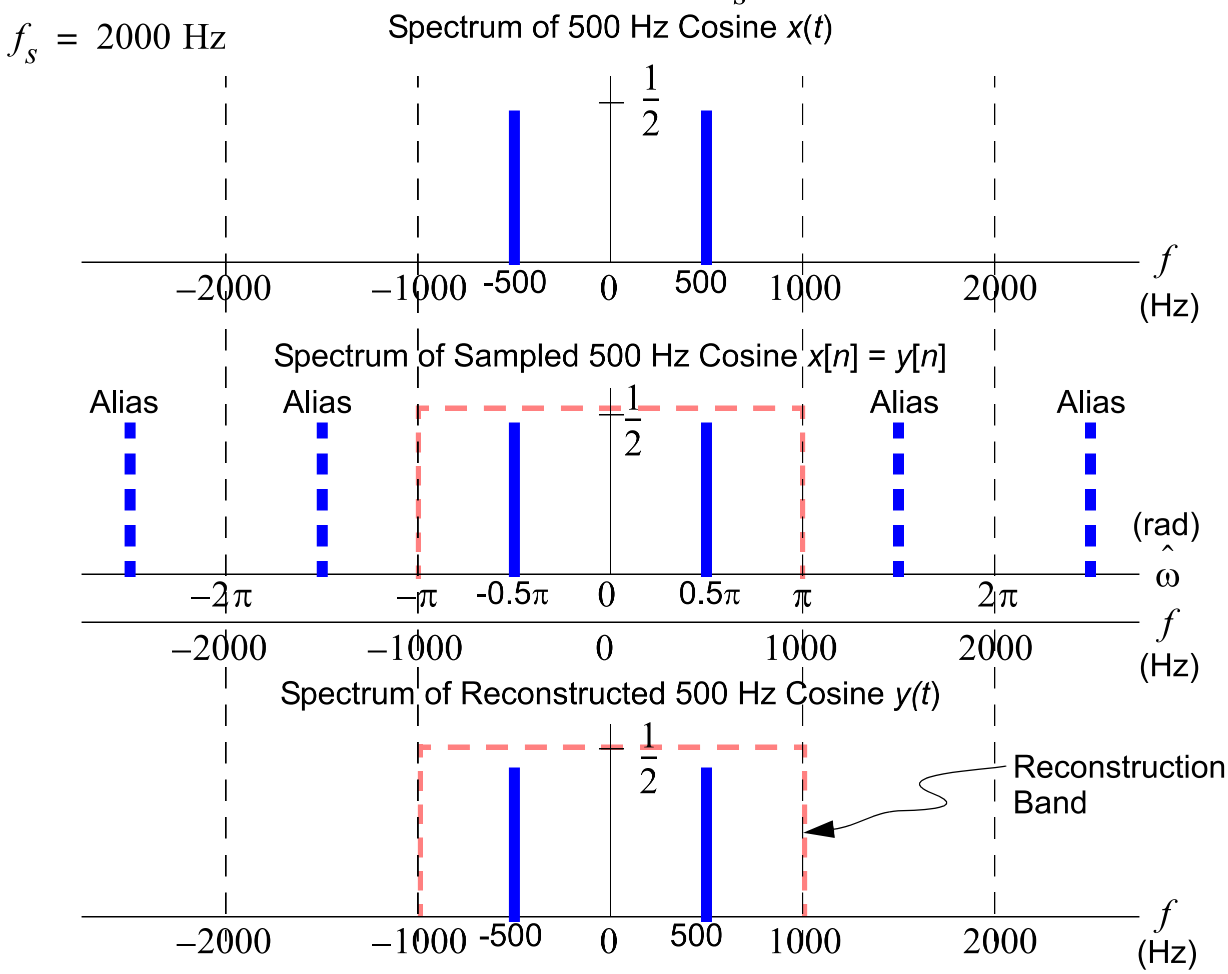


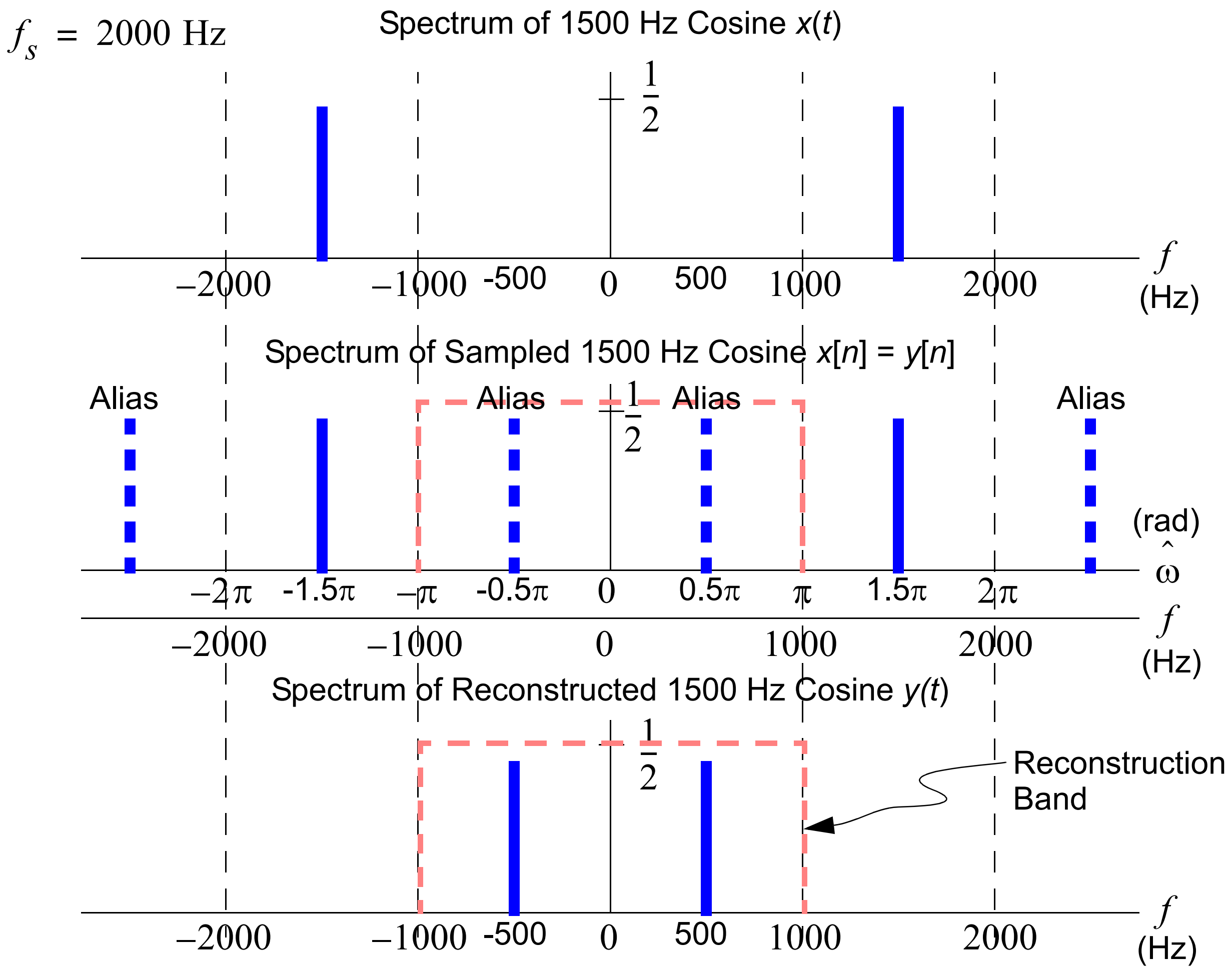
- With this system we can sample analog signal $x(t)$ to produce $x[n]$, and at the very least we may pass $x[n]$ directly to $y[n]$, then reconstruct the samples $y[n]$ into $y(t)$
  - The DSP system that sits between the C-to-D and D-to-C, should do something useful, but as a starting point we consider how well a *direct connection system* does at returning $y(t) \cong x(t)$
  - As long as the sampling theorem is satisfied, we expect that $y(t)$ will be close to $x(t)$ for frequency content in $x(t)$ that is less than $f_s/2$ Hz
  - What if some of the signals contained in $x(t)$ do not satisfy the sampling theorem?
  - Typically the C-to-D is designed to block signals above $f_s/2$ from entering the C-to-D (*antialiasing filter*)
  - A practical D-to-C is designed to reconstruct the principle alias frequencies that span

$$\hat{\omega} \in [-\pi, \pi] \Leftrightarrow f \in [-f_s/2, f_s/2] \tag{4.17}$$

Spectrum View of Sampling and Reconstruction

- We now view the spectra associated with a cosine signal passing through a C-to-D/D-to-C system
- Assume that $x(t) = \cos(2\pi f_0 t)$
- The sampling rate will be fixed at $f_s = 2000\,\text{Hz}$





- We see that the 1500 Hz sinusoid is aliased to 500 Hz, and when it is output as $y(t)$, we have no idea that it arrived at the input as 1500 Hz

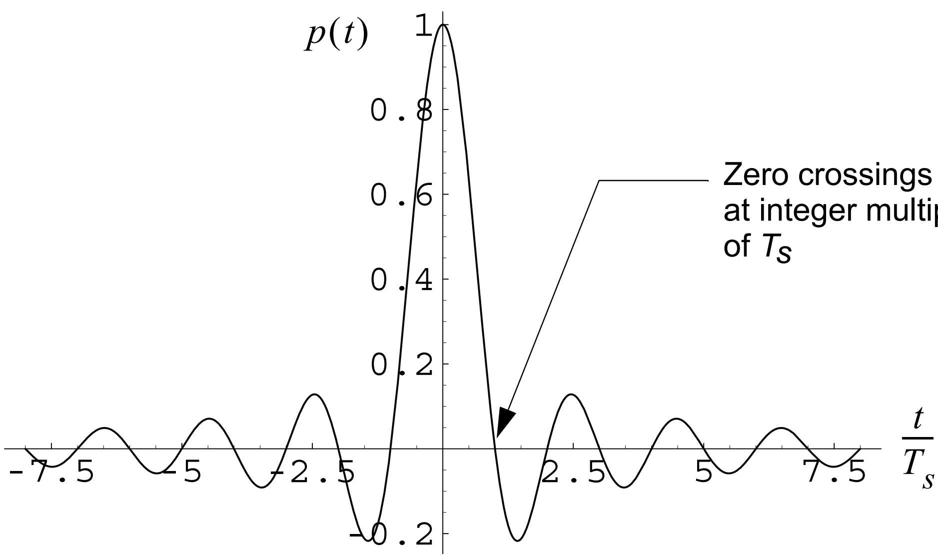
- What are some other inputs that will produce a 500 Hz output?

The Ideal Bandlimited Interpolation

- In Chapter 12 of the text it is shown that ideal D-to-C conversion utilizes an interpolating pulse shape of the form

$$p(t) = \frac{\sin(\pi t/T_s)}{(\pi t/T_s)}, -\infty < t < \infty \tag{4.18}$$

- The function $\sin(\pi x)/(\pi x)$ is known as the sinc function
- Note that interpolation with this function means that all samples are required to reconstruct $y(t)$, since the extent of $p(t)$ is doubly infinite
  – In practice this form of reconstruction is not possible



- A Mathematica animation showing that when the sinc() pulses are weighted by the sample values, delayed, and then summed, high quality reconstruction (interpolation) is possible
  – The code used to create the animation

```
Manipulate[
 Show[Plot[Cos[2 π f t + ϕ], {t, 0, 10}, PlotStyle → {Dashing[0.01], RGBColor[1, 0, 0]}],
  Plot[Sum[Cos[2 π f n + ϕ] Sinc[π (t - n)], {n, 0, 10}], {t, 0, 10},
   PlotStyle → {Thick, RGBColor[0, 1, 0]}], DiscretePlot[Cos[2 π f n + ϕ],
   {n, 0, 10}, Filling → Axis, PlotStyle → {PointSize[.02], RGBColor[1, 0, 0]}],
  Plot[Table[Cos[2 π f n + ϕ] Sinc[π (t - n)], {n, 0, 10}], {t, 0, 10}, PlotRange → All],
  PlotRange → All],
 {{f, 1 / 10}, {1 / 2, 1 / 3, 1 / 4, 1 / 10, 1 / 20}}, {{ϕ, 0}, {0, π / 4, π / 2}}]
```

- The final display showing an interpolated output for a single sinusoid
  - The input signal is $x(t) = \cos(2\pi f t + \phi)$ and we assume that $T_s = 1$, so in sampling we let $t \to n$
  - With $f = 1/4$ (4 samples per period) and $\phi = \pi/4$ we have the following display: