IIR Filters

Chapter 8

In this chapter we finally study the general infinite impulse response (IIR) difference equation that was mentioned back in Chapter 5. The filters will now include both feedback and feedforward terms. The system function will be a rational function where in general both the zeros and the poles are at nonzero locations in the z-plane.

The General IIR Difference Equation

- The general IIR difference equation described in Chapter 5 was of the form

$$\sum_{l=0}^{N} a_l y[n-l] = \sum_{k=0}^{M} b_k x[n-k] \tag{8.1}$$

- In this chapter the text rearranges this equation so that $y[n]$ is on the left and all of the other terms are on the right

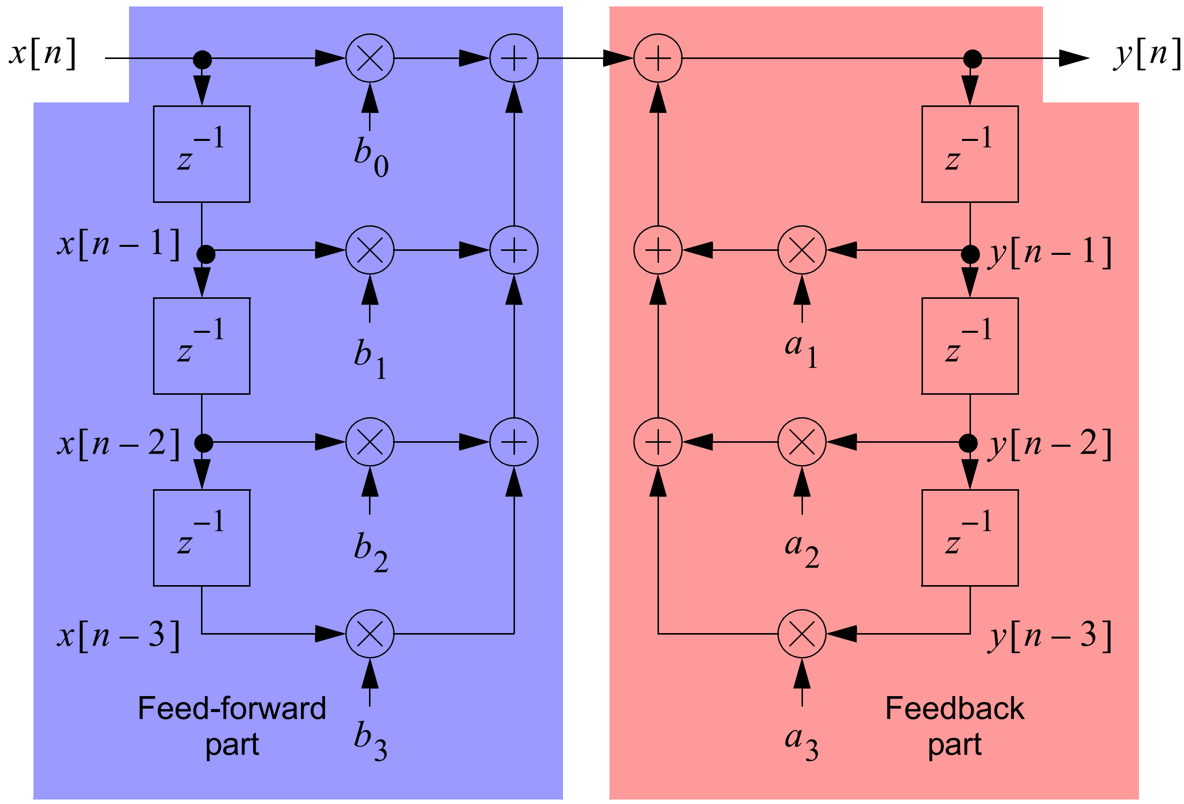
$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \tag{8.2}$$

- In so doing notice the sign change of the $\{a_l\}$ coefficients, and also we assume that $a_0 = 1$
- The total coefficient count is $N + M + 1$, meaning that this many multiplies are needed to compute each new output from the difference equation

Block Diagram

- As a special case consider $N = M = 3$

$$y[n] = \sum_{l=1}^{3} a_l y[n-l] + \sum_{k=0}^{3} b_k x[n-k]$$



Direct Form I Structure

- The above logically extends to any order
- This structure is known as *Direct-Form I*

Time-Domain Response

- To get started with IIR time-domain analysis we will consider a first-order filter ($N = 1$) with $M = 0$

$$y[n] = a_1 y[n-1] + b_0 x[n] \tag{8.3}$$

Impulse Response of a First-Order IIR System

- The impulse response can be obtained by setting $x[n] = \delta[n]$ and insuring that the system is *initially at rest*
- **Definition**: Initial rest conditions for an IIR filter means that:
  - (1) The input is zero prior to the start time n_0, that is $x[n] = 0$ for $n < n_0$
  - (2) The output is zero prior to the start time, that is $y[n] = 0$ for $n < n_0$
- We now proceed to find the impulse response of (8.3) via <u>direct recursion</u> of the difference equation

$$\begin{aligned} y[0] &= a_1 \cancelto{0}{y[-1]} + b_0\delta[0] = b_0 \\ y[1] &= a_1 y[0] + \cancelto{0}{b_0\delta[1]} = a_1 b_0 \\ y[2] &= a_1 y[1] + \cancelto{0}{b_0\delta[2]} = a_1^2 b_0 \\ &\dots \\ y[n] &= a_1^n b_0,\ n \ge 0 \end{aligned}$$

- In summary we have shown that the impulse response of a 1st-order IIR filter is

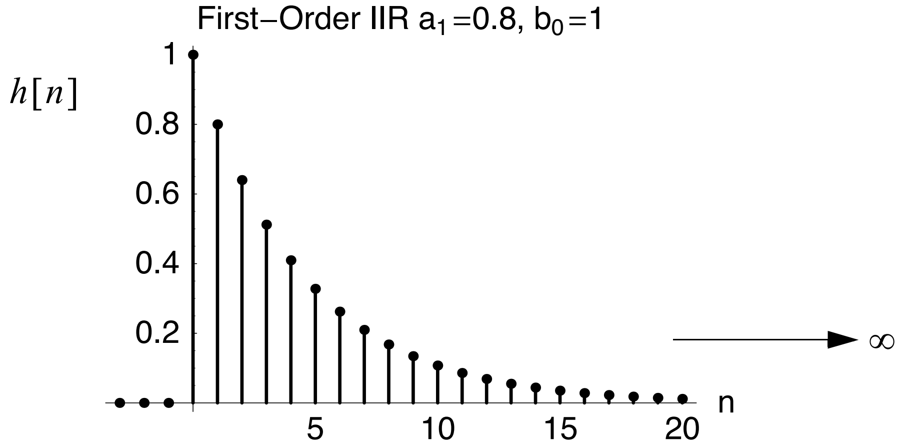
$$h[n] = b_0 (a_1)^n u[n] \tag{8.4}$$

where the unit step $u[n]$ has been utilized to make it clear that the output is zero for $n < 0$

Example: First-Order IIR with $b_0 = 1, a_1 = 0.8$

- The impulse response is

$$h[n] = 0.8^n u[n]$$



Linearity and Time Invariance of IIR Filters

- Recall that in Chapter 5 the definitions of time invariance and linearity were introduced and shown to hold for FIR filters
- It can be shown that the general IIR difference equation also exhibits linearity and time invariance
- Using linearity and time invariance we can find the output of the first-order system to a linear combination of time shifted impulses

$$x[n] = \sum_{k = N_1}^{N_2} x[k]\delta[n-k] \tag{8.5}$$

- From the impulse response of (8.4)

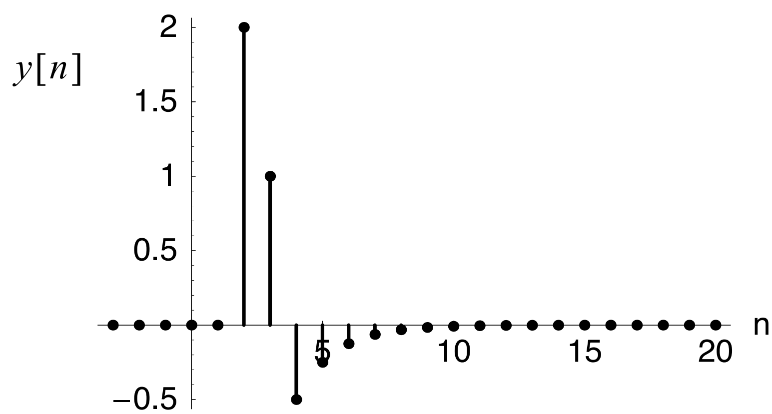
$$y[n] = \sum_{k=N_1}^{N_2} x[k]h[n-k]$$
$$= \sum_{k=N_1}^{N_2} x[k]\{b_0(a_1)^{n-k}u[n-k]\} \tag{8.6}$$

Example: $x[n] = 2\delta[n-2] - \delta[n-4]$, $a_1 = 0.5$ and $b_0 = 1$

- Using the above result, it follows that

$$y[n] = 2(0.5)^{n-2}u[n-2] - (0.5)^{n-4}u[n-4]$$

- Plotting this function results in



- Linearity and time-invariance can also be used to find the impulse response of related IIR filters, e.g.,

$$y[n] = a_1y[n-1] + b_0x[n] + b_1x[n-1] \tag{8.7}$$

- We can view this as the superposition of an undelayed and delayed input to the filter

$$y[n] = a_1y[n-1] + x[n] \tag{8.8}$$

- Based on this observation, the impulse response is

$$\begin{aligned} h[n] &= b_0(a_1)^n u[n] + b_1(a_1)^{n-1} u[n-1] \\ &= b_0\delta[n] + (b_0 + b_1 a_1^{-1})(a_1)^n u[n-1] \end{aligned} \tag{8.9}$$

Step Response of a First-Order Recursive System

- The step response allows us to see how a filter (system) responds to an infinitely long input
- We now consider the step response of

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

- Via direct recursion of the difference equation

$$\begin{aligned} y[0] &= a_1 \overset{0}{y[-1]} + b_0 u[0] = b_0 \\ y[1] &= a_1 y[0] + b_0 u[1] = a_1 b_0 + b_0 \\ y[2] &= a_1 y[1] + b_0 u[2] = a_1(a_1 b_0 + b_0) + b_0 \\ &\cdots \\ y[n] &= b_0(1 + a_1 + \cdots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k \end{aligned}$$

- The summary form indicates a finite geometric series, which has solution

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r}, & r \neq 1 \\ L + 1, & r = 1 \end{cases} \tag{8.10}$$

- Using (8.10) and assuming that $a_1 \neq 1$, the step response of the first-order filter is

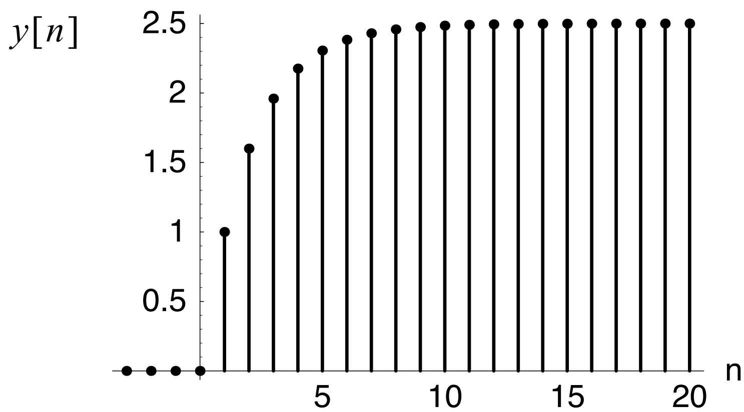
$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} u[n] \tag{8.11}$$

- Three conditions for a_1 exist
  1. When $|a_1| > 1$ the term a_1^{n+1} grows without bound as n becomes large, resulting in an **unstable condition**
  2. When $|a_1| < 1$ the a_1^{n+1} term decays to zero as $n \to \infty$, and we have a **stable condition**
  3. When $a_1 = 1$ we have the special case output of (8.10) where the output is of the form $b_0(n+1)$, which also grows without bound; with $a_1 = -1$ the output alternates sign, hence we have a **marginally stable condition**

<u>Example</u>: $a_1 = 0.6$, $b_0 = 1$ and $x[n] = u[n]$

- The step response of this filter is

$$y[n] = \frac{1 - (0.6)^{n+1}}{1 - 0.6} u[n]$$



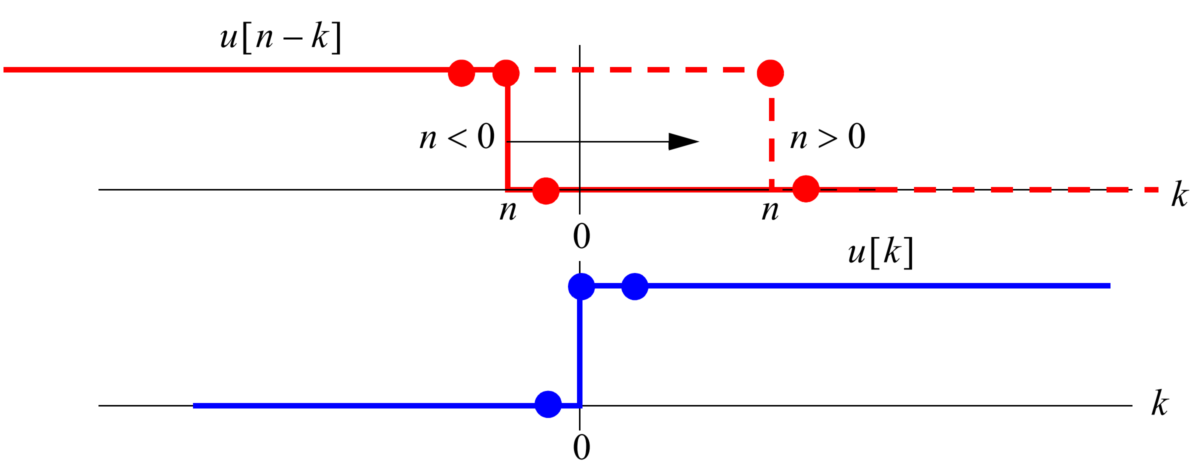
- The step response can also be obtained by direct evaluation of the convolution sum

$$y[n] = x[n]*h[n] = u[n]*h[n] \tag{8.4}$$

- For the problem at hand

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]\{b_0(a_1)^{n-k}u[n-k]\} \tag{8.5}$$

- To evaluate this requires careful attention to details
- The product $u[k]u[n-k]$ tells us how to set the sum limits



- The result is

$$\begin{aligned} y[n] &= \left(\sum_{k=0}^{n} b_0 (a_1)^{n-k}\right) u[n] \\ &= b_0 (a_1)^n \sum_{k=0}^{n} (a_1)^{-k} \\ &= b_0 (a_1)^n \frac{1-(1/a_1)^{n+1}}{1-(1/a_1)} \\ &= b_0 \frac{1-(a_1)^{n+1}}{1-(a_1)} \end{aligned} \tag{8.6}$$

which is the same result obtained by the direct recursion

System Function of an IIR Filter

- From our study of the z-transform we know that convolution in the time (sequence)-domain corresponds to multiplication in the z-domain

$$y[n] = x[n]*h[n] \overset{z}{\leftrightarrow} X(z)H(z) = Y(z)$$

- For the case of IIR filters $H(z)$ will be a fully rational function, meaning in general both poles and zeros (more than at $z = 0$)
- Begin by z-transforming both sides of the general IIR difference equation using the delay property

$$Y(z) = \sum_{l=1}^{N} a_l \overbrace{z^{-l} Y(z)}^{\text{ZT}\{y[n-l]\}} + \sum_{k=0}^{M} b_k \overbrace{z^{-k} X(z)}^{\text{ZT}\{x[n-k]\}} \tag{8.7}$$

- Form the ratio $Y(z)/X(z) = H(z)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}} \tag{8.8}$$

- The coefficients of the numerator polynomial, denoted $B(z)$, correspond to the feed-forward terms of the difference equation
- The coefficients of the denominator polynomial, denoted $A(z)$, for $z^{-l}, l > 0$ correspond to the feedback terms of the difference equation
- We have used various MATLAB functions that take as input `b` and `a` coefficient vectors, e.g., `filter(b,a,...)`, `freqz(b,a,...)`, and `zplane(b,a)`
- In terms of the general IIR system we now identify those vectors as

$$\begin{aligned} \texttt{b} &= [b_0, b_1, \ldots, b_M] \\ \texttt{a} &= [1, -a_1, -a_2, \ldots, -a_N] \end{aligned} \tag{8.9}$$

The General First-Order Case

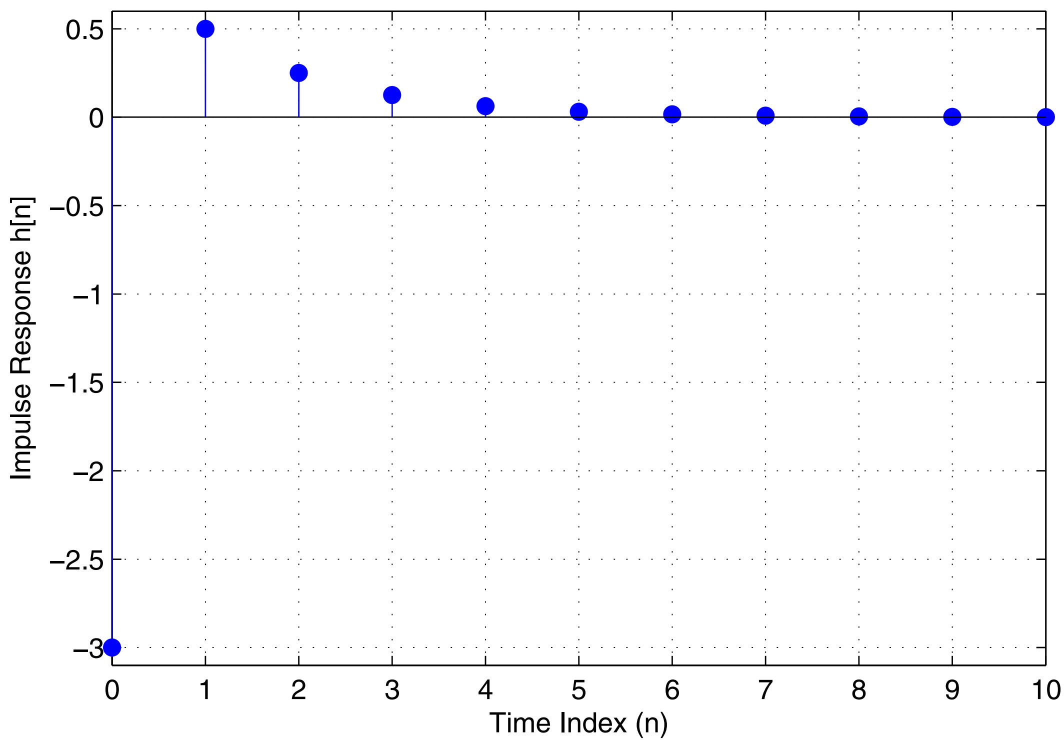
- As a special case consider $N = M = 1$, then

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \tag{8.10}$$

Example: Impulse Response Using MATLAB

- Suppose that $a_1 = 0.5$, $b_0 = -3$, and $b_1 = 2$

```
>> n = 0:20;
>> x = [1 zeros(1,20)]; % impulse sequence input
>> y = filter([-3,2],[1 -0.5],x);
>> stem(n,y,'filled')
>> axis([0 10 -3.1 .6])
>> grid
>> ylabel('Impulse Response h[n]')
>> xlabel('Time Index (n)')
```



Example: `y = filter([1 1],[1 -0.8],x)`

- We wish to find the system function, impulse response, and difference equation that corresponds to the given `filter()` expression
- By inspection

$$H(z) = \frac{1 + z^{-1}}{1 - 0.8z^{-1}}$$

- The impulse response using page 8–5, eqns (8.7)—(8.9)

$$h[n] = \delta[n] + (1 + 0.8^{-1})(0.8)^n u[n-1]$$

- The difference equation is

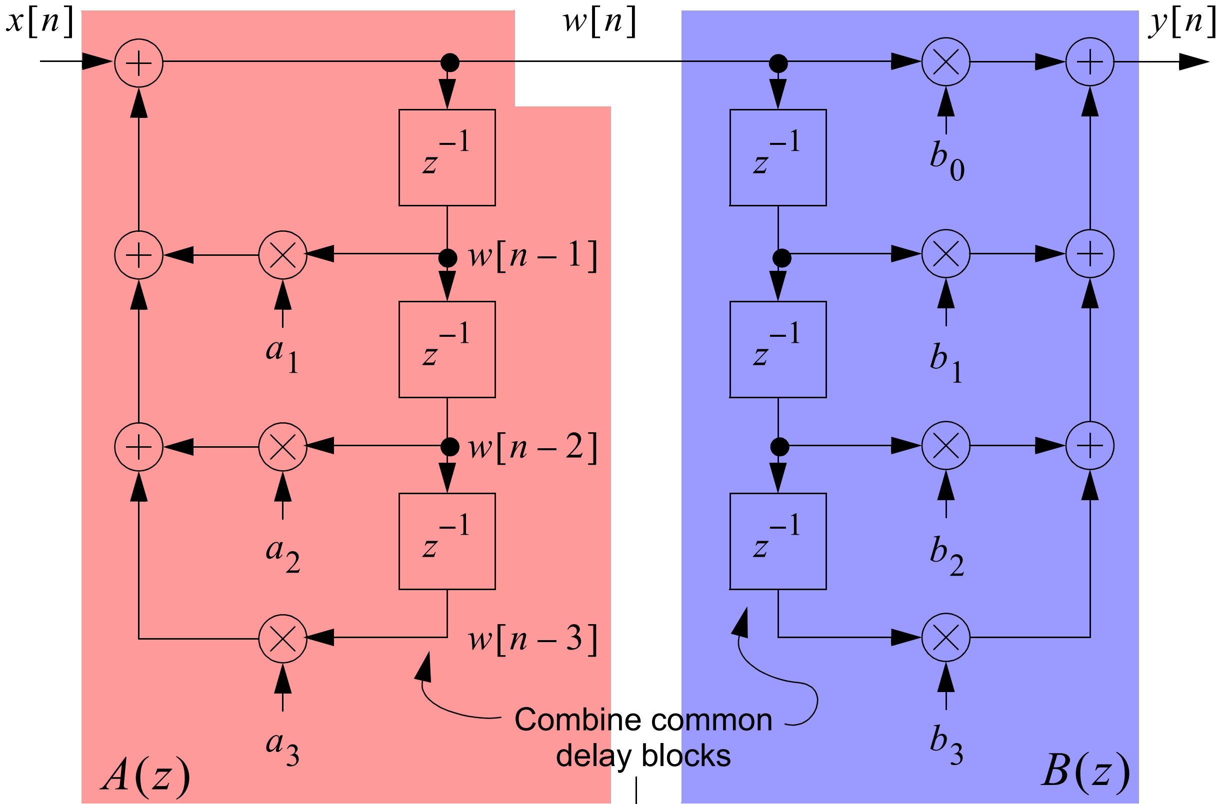
$$y[n] = 0.8y[n-1] + x[n] + x[n-1]$$

System Functions and Block-Diagram Structures

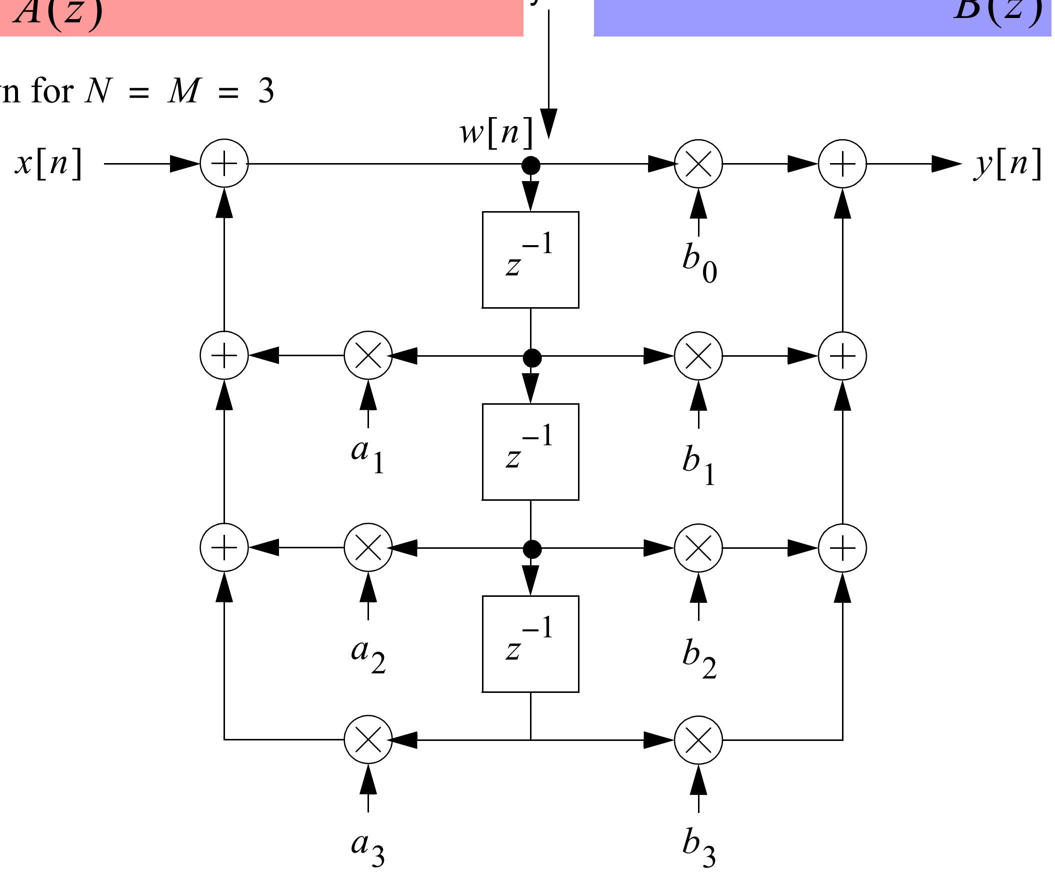
- We have already examined the Direct-Form I structure (p. 8–2)
- The Direct-Form I structure implements the feed-forward terms first followed by the feedback terms
- We can view this as a cascade of two systems, which due to linearity can also be written as

$$H(z) = B(z) \cdot \frac{1}{A(z)} = \frac{1}{A(z)} \cdot B(z) \tag{8.11}$$

- In the block diagram this is represented as



Shown for $N = M = 3$

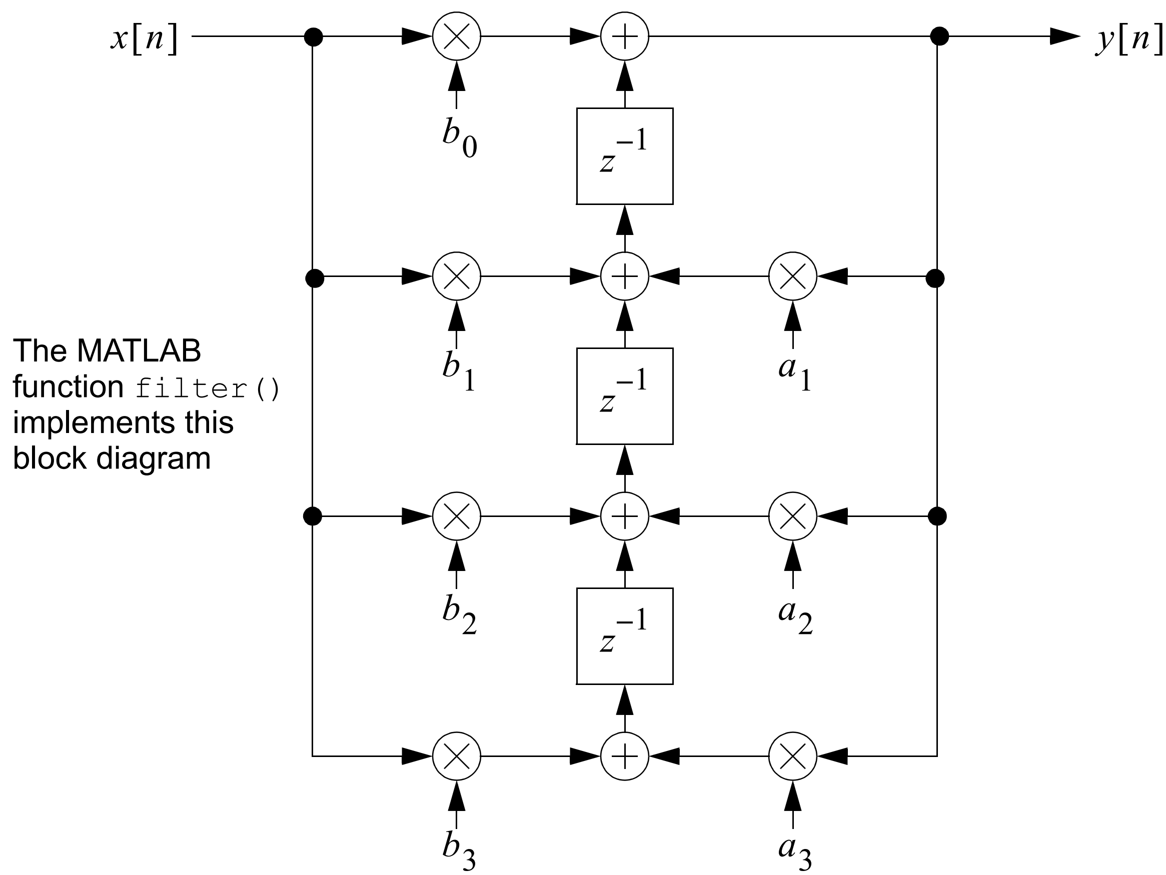


Direct-Form II

- The Direct-Form II structure uses fewer delay blocks than Direct-Form I

The Transposed Structures

- A property of filter block diagrams is that
  - When all of the arrows are reversed
  - All branch points become summing nodes; all summing nodes become branch points
  - The input and output are interchanged
  - The system function is unchanged



Transposed Direct-Form II

Relation to the Impulse Response

- From Chapter 7 we know that the impulse response and system function are related via the z-transform
- For IIR systems more work is required to obtain the z-transform
- Consider $y[n] = ay[n-1] + x[n]$, where we have learned that the impulse response is

$$h[n] = a^n u[n]$$

- From the definition of the z-transform,

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \tag{8.12}$$

- The sum of (8.12) is an infinite geometric series which in general terms is

$$S = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

- Applying the sum formula to (8.12) results in

$$H(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}, |z| > |a| \tag{8.13}$$

  – The condition that $|z| > |a|$ tells us that the z-transform only exists for these values of z
  – The z-plane region is known as the *region of convergence*
- We have thus established the following z-transform relation-

ship

$$\boxed{a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1-az^{-1}}} \tag{8.14}$$

- We can use this result to find the z-transform of

$$h[n] = b_0(a_1)^n u[n] + b_1(a_1)^{n-1} u[n-1] \tag{8.15}$$

directly using just linearity and the delay property

$$H(z) = b_0 \frac{1}{1-a_1 z^{-1}} + b_1 z^{-1} \frac{1}{1-a_1 z^{-1}}$$

$$= \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}}$$

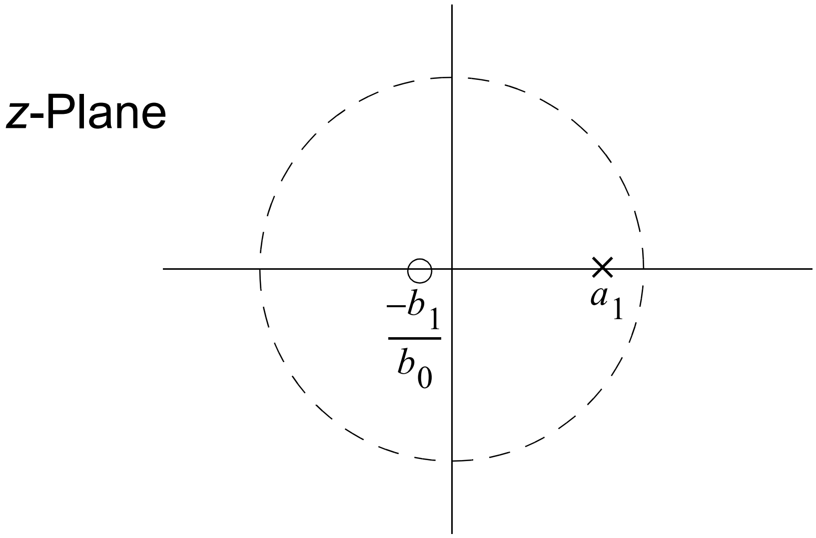
- We will learn later that we can work this operation in reverse, and when combined with *partial fraction expansion*, we will be able to find the inverse z-transform of almost any rational $H(z)$

Poles and Zeros

- Factoring the numerator denominator polynomials allows us to discover the poles and zeros of $H(z)$
- For the case of a first-order system only algebra is needed

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}} \cdot \frac{z}{z} = \frac{b_0 z + b_1}{z - a_1} = b_0 \frac{z + b_1/b_0}{z - a_1}$$

- The single pole and zero are $p_1 = a_1$ and $z_1 = -b_1/b_0$



Poles or Zeros at the Origin or Infinity

- For the general IIR filter/system the number of poles always equals the number of zeros
- For FIR systems we saw that all of the poles were at $z = 0$
- It is also possible to have poles or zeros at $z = \infty$

Example: Zero at $z = \infty$

- Consider

$$H(z) = \frac{2z^{-1}}{1 - 0.8z^{-1}} = \frac{2}{z - 0.8}$$

- This system has a pole at $z = 0.8$ and zero at $z = \infty$ since $\lim_{z \to \infty} H(z) = 0$

Example: Pole at $z = \infty$

- Consider

$$H(z) = \frac{1 + 0.5z^{-1}}{z^{-1}} = z + 0.5$$

- This system has a pole at $z = \infty$ and a zero at $z = -0.5$

Pole Locations and Stability

- We know that

$$h[n] = a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}} = H(z) \tag{8.16}$$

- We note that this system has a pole at $z = a$ and a zero at $z = 0$
- The impulse response decays to zero so long as $|a| < 1$, which is equivalent to requiring that the pole lies inside the unit circle
- **System Stability**: Causal LTI IIR systems, initially at rest, are stable if all of the poles of the system function lie inside the unit circle

Example: $H(z) = (1 - 5z^{-1})/(1 - 0.995z^{-1})$

- Converting to positive powers of z

$$H(z) = \frac{z - 5}{z - 0.995} \rightarrow \text{Pole at } z = 0.995 \text{ so stable}$$

Example: Second-order $H(z)$

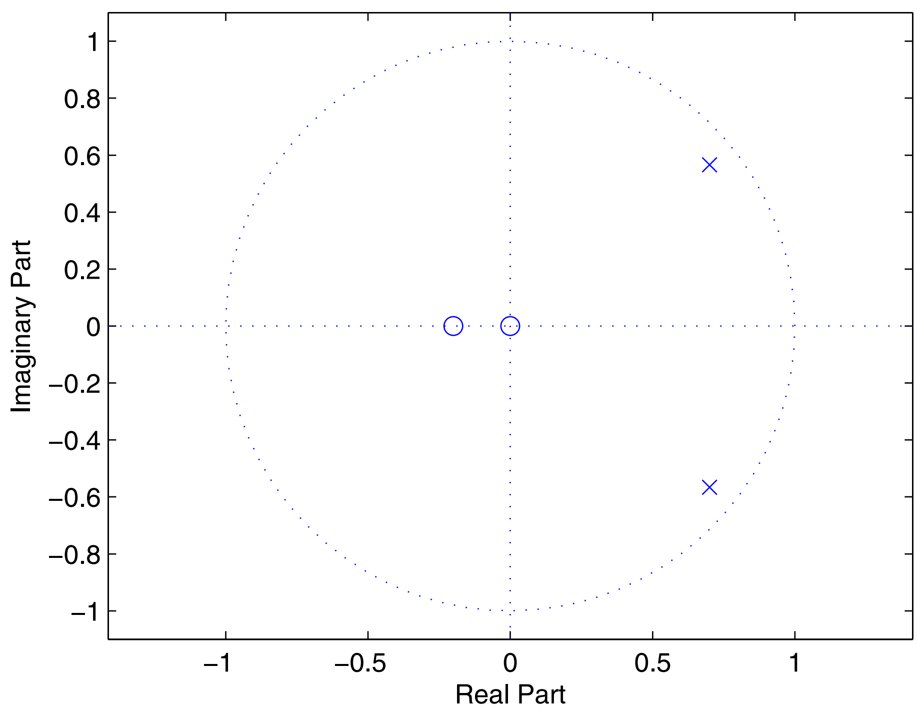
- Suppose that

$$H(z) = \frac{1 + 0.2z^{-1}}{1 - 1.4z^{-1} + 0.81z^{-2}} = \frac{z(z + 0.2)}{z^2 - 1.4z + 0.81}$$

$$= \frac{z(z + 0.2)}{(z - (0.7 + j0.4\sqrt{2}))(z - (0.7 - j0.4\sqrt{2}))}$$

- In polar form the poles are $p_{1,2} = 0.9e^{\pm j0.680}$, so the poles are inside the unit circle and the system is stable
- We can check stability using `zplane()` to plot the poles and zeros for us

```
>>> zplane([1 0.2],[1 -1.4 0.81])
```



Frequency Response of an IIR Filter

- From Chapter 7 we know that the frequency response is found by letting $z \to e^{j\hat{\omega}}$ in the system function (provided the system is stable)

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z = e^{j\hat{\omega}}} \tag{8.17}$$

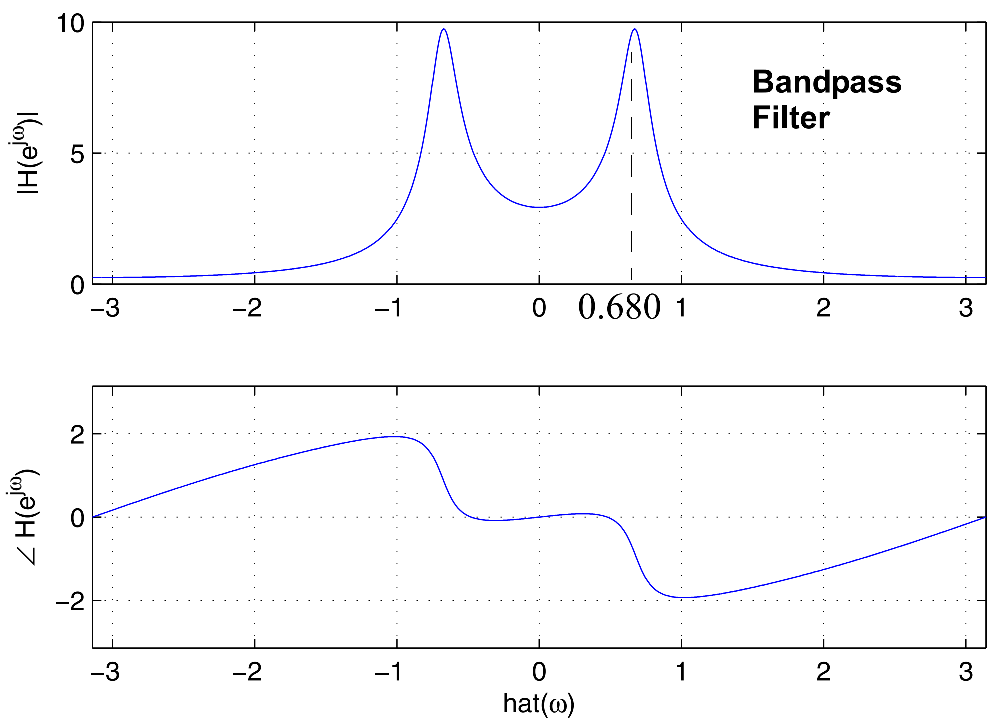
Example: $H(z) = (1 + 0.2z^{-1})/(1 - 1.4z^{-1} + 0.81z^{-2})$

- Making the substitution $z = e^{j\hat{\omega}}$ we have

$$H(e^{j\hat{\omega}}) = \frac{1 + 0.2e^{-j\hat{\omega}}}{1 - 1.4e^{-j\hat{\omega}} + 0.81e^{-j2\hat{\omega}}}$$

- We can use `freqz()` to plot the magnitude and phase

```
>> w = -pi:(pi/500):pi;
>> H = freqz([1 .2],[1 -1.4 0.81],w);
```



- This particular filter is a bandpass filter because it has a relative large magnitude response over a narrow band of frequencies and small response otherwise
- From the earlier pole-zero analysis, the peak gain is near the angle the poles make to the real axis, $\omega_0 = \pm 0.680$

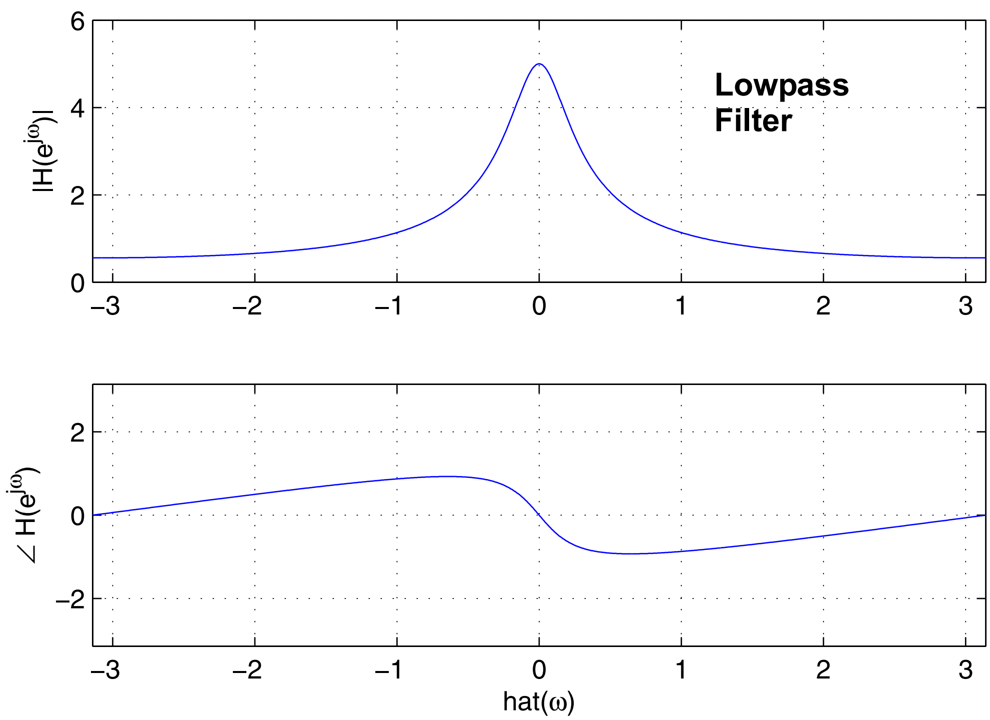
Example: $H(z) = 1/(1 - 0.8z^{-1})$

- Here we have

$$H(e^{j\hat{\omega}}) = \frac{1}{1 - 0.8e^{-j\hat{\omega}}}$$

Plotting using `freqz()` we have

```
>> w = -pi:(pi/500):pi;
>> H = freqz(1,[1 -0.8],w);
```



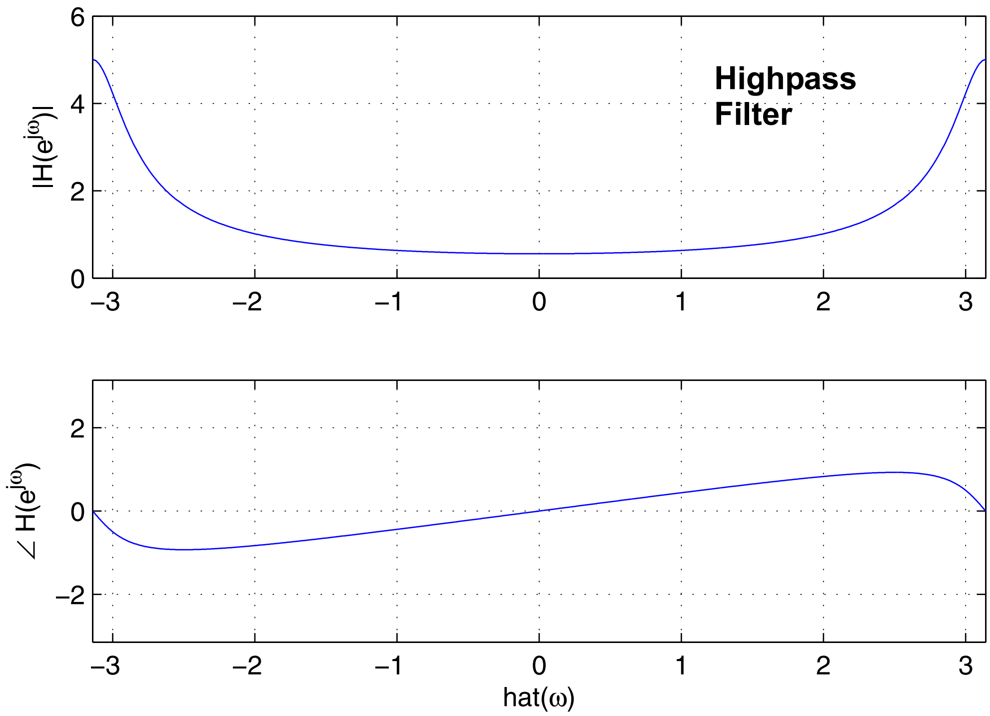
Example: $H(z) = 1/(1 + 0.8z^{-1})$

- Here we have

$$H(e^{j\hat{\omega}}) = \frac{1}{1 + 0.8e^{-j\hat{\omega}}}$$

Plotting using `freqz()` we have

```
>> w = -pi:(pi/500):pi;
>> H = freqz(1,[1 0.8],w);
```



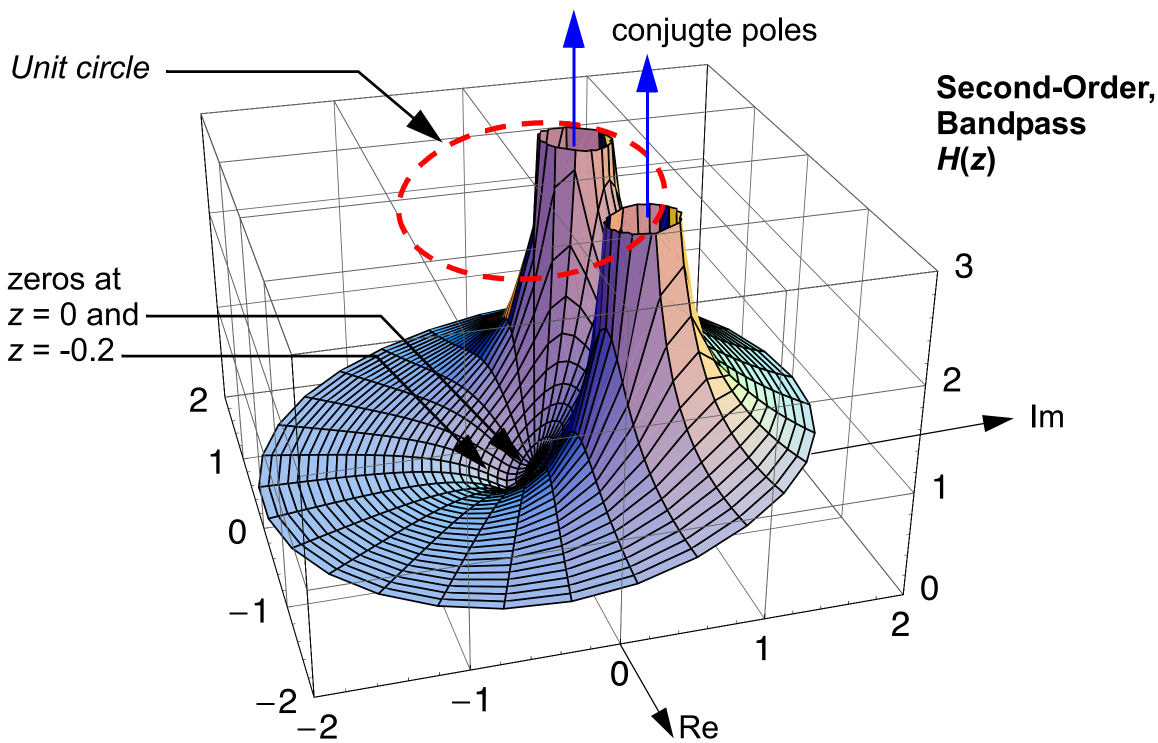
- Note that by just changing the sign of the coefficient a_1 the filter changes from being lowpass to highpass
- A similar behavior was found for the first-order FIR filter

3D Surface Plot of $|H(z)|$

- Consider the second-order system

$$H(z) = \frac{1 + 0.2z^{-1}}{1 - 1.4z^{-1} + 0.81z^{-2}}$$

- A 3D surface plot of $|H(z)|$ can help clarify how the frequency response is obtained by evaluating $H(z)$ around the unit circle



The Inverse z-Transform and Applications

- Finding the impulse response of a first-order IIR system was not too difficult using difference equation recursion, but for system order $N > 1$, this process becomes too difficult

- We need an inverse z-transform approach that will allow us to work from any rational system function, $H(z)$, backwards to $h[n]$
- Useful z-transform properties and pairs available to help this cause are listed below

z-Transform Relations

	$x[n]$	$\overset{z}{\leftrightarrow}$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\overset{z}{\leftrightarrow}$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\overset{z}{\leftrightarrow}$	$z^{-n_0}X(z)$
3.	$y[n] = x[n]*h[n]$	$\overset{z}{\leftrightarrow}$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\overset{z}{\leftrightarrow}$	1
5.	$\delta[n - n_0]$	$\overset{z}{\leftrightarrow}$	z^{-n_0}
6.	$a^n u[n]$	$\overset{z}{\leftrightarrow}$	$\dfrac{1}{1 - az^{-1}}$

A General Procedure for Inverse z-Transformation

- The technique we develop here uses an algebraic decomposition known as *partial fraction expansion*
- The function we wish to inverse transform, $H(z)$, is assumed for now to be *proper rational*, meaning that $M < N$

- **Step 1**: Factor the denominator polynomial into pole factors of the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$
- **Step 2**: Create a partial fraction expansion of $H(z)$ via

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

where $A_k = H(z)(1 - p_k z^{-1})\Big|_{z = p_k}$

- **Step 3**: The inverse z-transform via relation #6 is

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

- The limitation of this approach is that the p_k are distinct
  - In general there may be repeated poles, in which case the partial fraction expansion takes a slightly different form from step 2
  - Hence, at present we will only consider non-repeated poles

---

Example: $M = 1, N = 2$

$$H(z) = \frac{1 + 2z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

- First we factor the denominator

$$p_{1,2} = \frac{(3/4) \pm \sqrt{9/16 - 1/2}}{2 \cdot 1} = \frac{3}{8} \pm \frac{1}{8}$$

$$H(z) = \frac{1 + 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - \frac{1}{4}z^{-1}}$$

- Solving for A_1

$$A_1 = \left.\frac{1 + 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}\left(1 - \frac{1}{2}z^{-1}\right)\right|_{z = 1/2}$$

$$= \left.\frac{1 + 2z^{-1}}{1 - \frac{1}{4}z^{-1}}\right|_{z^{-1} = 2} = \frac{1 + 4}{1 - \frac{1}{2}} = 10$$

- Solving for A_2

$$A_2 = \left.\frac{1 + 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}\left(1 - \frac{1}{4}z^{-1}\right)\right|_{z = 1/4}$$

$$= \left.\frac{1 + 2z^{-1}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1} = 4} = \frac{1 + 8}{1 - 2} = -9$$

- So,

$$H(z) = \frac{10}{1 - \frac{1}{2}z^{-1}} - \frac{9}{1 - \frac{1}{4}z^{-1}}$$

- Inverse z-transform term-by-term using #6

$$h[n] = 10\left(\frac{1}{2}\right)^n u[n] - 9\left(\frac{1}{4}\right)^n u[n]$$

- The MATLAB signal processing toolbox has a function that can perform partial fraction expansion

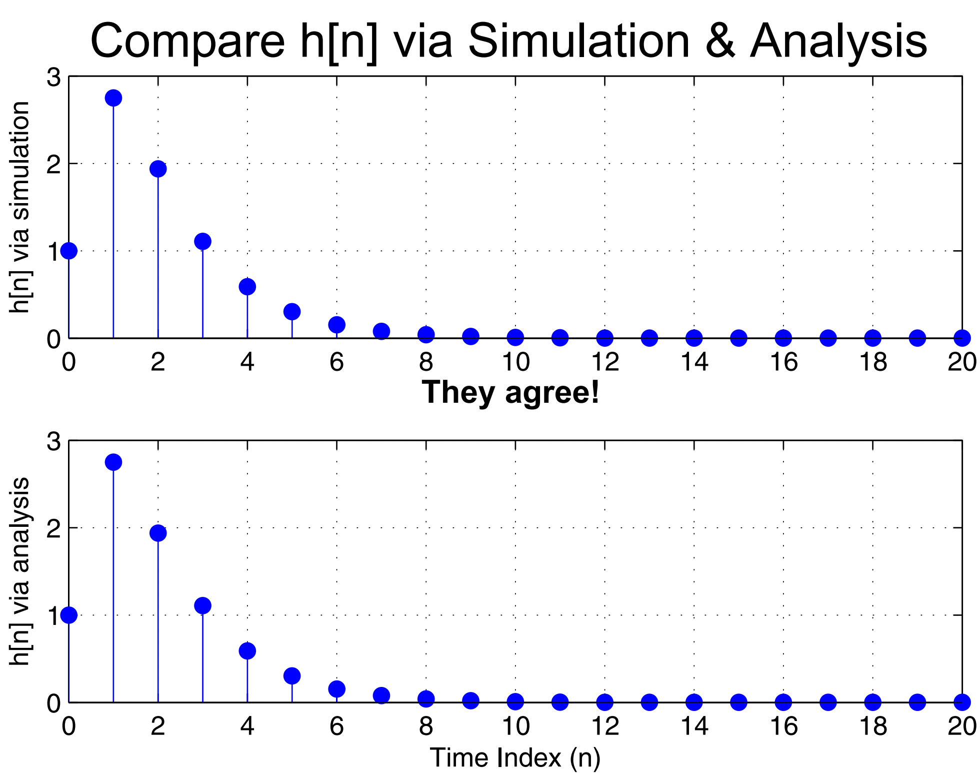
```
>> help residuez
 RESIDUEZ Z-transform partial-fraction expansion.
    [R,P,K] = RESIDUEZ(B,A) finds the residues, poles and direct terms
    of the partial-fraction expansion of B(z)/A(z),

       B(z)       r(1)               r(n)
       ---- = ------------ +...  ------------ + k(1) + k(2)z^(-1) ...
       A(z)   1-p(1)z^(-1)       1-p(n)z^(-1)

    B and A are the numerator and denominator polynomial coefficients,
    respectively, in ascending powers of z^(-1).  R and P are column
    vectors containing the residues and poles, respectively.  K contains
    the direct terms in a row vector.  The number of poles is
       n = length(A)-1 = length(R) = length(P)
    The direct term coefficient vector is empty if length(B) < length(A);
    otherwise,
       length(K) = length(B)-length(A)+1

    If P(j) = ... = P(j+m-1) is a pole of multiplicity m, then the
    expansion includes terms of the form
            R(j)              R(j+1)                       R(j+m-1)
       -------------- + ------------------   + ... + ------------------
       1 - P(j)z^(-1)   (1 - P(j)z^(-1))^2           (1 - P(j)z^(-1))^m

    [B,A] = RESIDUEZ(R,P,K) converts the partial-fraction expansion back
    to B/A form.
```

- Using `residuez()` we find:

```
>> [A,p,K] = residuez([1 2],[1 -3/4 1/8])

A = 10          % The partial fraction coefficients
    -9          % agree
p = 5.0000e-01  % The pole factoring agrees
    2.5000e-01  %
K = []          % Results from long division
                % to make proper rational (NA here).
```

- As a further check we can plot $h[n]$ directly and compare it to the results obtained by direct evaluation of the difference equation via `filter()`

```
n = 0:20;
x = [1 zeros(1,20)];
y = filter([1 2],[1 -3/4 1/8],x);
h = 10*(1/2).^n -9*(1/4).^n;
subplot(211)
stem(n,y,'filled')
grid
subplot(212)
stem(n,h,'filled')
grid
```



Example: $y[n] = x[n]*h[n]$

- Find $y[n]$ for an IIR system having input $x[n] = 2u[n]$ and system function

$$H(z) = \frac{1 + z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

- We will first find $Y(z)$
- From table entry #6 with $a = 1$

$$X(z) = \frac{2}{1 - z^{-1}}$$

- As a result of table entry #3

$$Y(z) = X(z)H(z) = \frac{2 + 2z^{-1}}{(1 - z^{-1})\left(1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right)}$$

$$= \frac{2 + 2z^{-1}}{(1 - z^{-1})\left(1 + \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

- We now use a partial fraction expansion over the three real poles 1, -1/3, and 1/2

$$Y(z) = \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 + \frac{1}{3}z^{-1}} + \frac{A_3}{1 - \frac{1}{2}z^{-1}}$$

- Solving for the coefficients

$$A_1 = \left.\frac{2+2z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=1} = \frac{2+2}{\frac{4}{3}\cdot\frac{1}{2}} = 6$$

$$A_2 = \left.\frac{2+2z^{-1}}{(1-z^{-1})\left(1-\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=-3} = \frac{2-6}{4\cdot\frac{5}{2}} = -\frac{2}{5}$$

$$A_3 = \left.\frac{2+2z^{-1}}{(1-z^{-1})\left(1+\frac{1}{3}z^{-1}\right)}\right|_{z^{-1}=2} = \frac{2+4}{-1\cdot\frac{5}{3}} = -\frac{18}{5}$$

- Finally,

$$Y(z) = \frac{6}{1-z^{-1}} - \frac{2/5}{1+\frac{1}{3}z^{-1}} - \frac{18/5}{1-\frac{1}{2}z^{-1}}$$

and using #6 to inverse transform term-by-term

$$y[n] = 6u[n] - \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n] - \frac{18}{5}\left(\frac{1}{2}\right)^n u[n]$$

We can check this result using `residuez()`

```
>> [A,p,K] = residuez([2 2],conv([1 -1],[1 -1/6 -1/6]))

A = 6.0000e+00  <== agrees with A1 = 6
   -3.6000e+00  <== agrees with A3 = -18/5
   -4.0000e-01  <== agrees with A2 = -2/5
```

```
p = 1.0000e+00
    5.0000e-01
   -3.3333e-01

K = []
```

- The results agree!

- The partial fraction expansion technique **requires** that $M < N$
- If the rational function does not satisfy this condition we can perform long division to reduce the order of the denominator to the point where $M < N$

<u>Example</u>: Long Division

- Consider

$$Y(z) = \frac{2 - 2.4z^{-1} - 0.4z^{-2}}{1 - 0.3z^{-1} - 0.4z^{-2}}$$

where $N = M = 2$ (not proper rational)

- Perform long division

$$\begin{array}{r} 1 \phantom{- 2.4z^{-1} - 0.4z^{-2}} \\ 1 - 0.3z^{-1} - 0.4z^{-2} \overline{\big) 2 - 2.4z^{-1} - 0.4z^{-2}} \\ \underline{1 - 0.3z^{-1} - 0.4z^{-2}} \\ 1 - 2.1z^{-1} \phantom{- 0.4z^{-2}} \end{array}$$

- We now have reduced $Y(z)$ to the form

$$Y(z) = 1 + \frac{1 - 2.1z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}}$$

- We can now perform a partial fraction expansion on the rational function

$$Y'(z) = \frac{1-2.1z^{-1}}{1-0.3z^{-1}-0.4z^{-2}} = \frac{1-2.1z^{-1}}{(1+0.5z^{-1})(1-0.8z^{-1})}$$

$$= \frac{A_1}{1+0.5z^{-1}} + \frac{A_2}{1-0.8z^{-1}}$$

- The coefficients are

$$A_1 = \left.\frac{1-2.1z^{-1}}{1-0.8z^{-1}}\right|_{z^{-1}=-2} = \frac{1+4.2}{1+1.6} = 2$$

$$A_2 = \left.\frac{1-2.1z^{-1}}{1+0.5z^{-1}}\right|_{z^{-1}=1.25} = \frac{1-2.625}{1+0.625} = -1$$

- Finally,

$$Y(z) = 1 + \frac{2}{1+0.5z^{-1}} - \frac{1}{1-0.8z^{-1}}$$

and

$$y[n] = \delta[n] + 2(-0.5)^n u[n] - (0.8)^n u[n]$$

- We can again check this with `residuez()`; which will automatically perform long division

```
>> [A,p,K] = residuez([2 -2.4 -0.4],[1 -0.3 -0.4])

A = -1  <== agrees with A2 = -1
     2  <== agrees with A1 = 2
```

```
p = 8.0000e-01
   -5.0000e-01

K = 1 <== Long division term
```

- The value $K = 1$ is the result of long division (see the help for `residuez()`)
- The answers agree!

Steady-State Response and Stability

- The sinusoidal steady-state response developed in Chapter 6 for FIR filters also holds for IIR filters
- It can be shown (see Text Section 8-8 for more details, especially for $N = 1$) that for

$$x[n] = e^{j\hat{\omega}_0 n}u[n] \tag{8.18}$$

and $H(z)$ an IIR system with $N > M$, the system output will be

$$y[n] = \sum_{k=1}^{N} A_k(p_k)^n u[n] + H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n}u[n] \tag{8.19}$$

where p_k, $k = 1, \ldots, N$ are the poles of $H(z)$ and the A_k are the corresponding partial fraction coefficients

- The first term represents the transient response, which provided all the poles of $H(z)$ lie inside the unit circle, will decay to zero, leaving just the sinusoidal term

- For the output to reach sinusoidal steady-state we must have $|p_k| < 1$ to insure that the transient term (first term) decays to zero, this then insures that the system is stable
- **Summary**: Poles inside the unit circle to insure stability

---

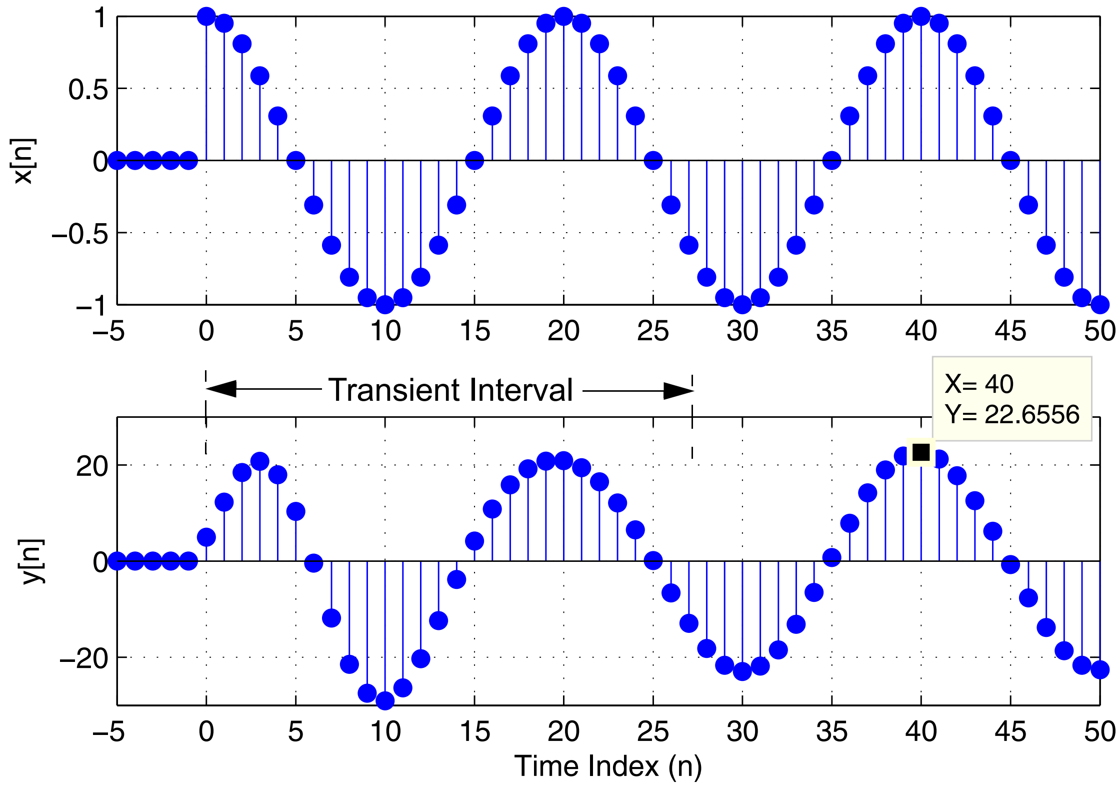
Example: $x[n] = \cos((2\pi/20)n)u[n]$

- Input a cosine with $\hat{\omega}_0 = 0.1\pi$ starting at $n = 0$ to the system

$$H(z) = \frac{5}{1 - 1.8z^{-1} + 0.9z^{-2}}$$

- This is a second-order IIR filter, so the transient term consists of two exponentials
- Second-order system will be studied in more detail in the next section
- Observe the transient using MATLAB

```
>> n = -5:50;
>> x = cos(2*pi/20*n).*ustep(n,0);
>> y = filter(5,[1 -1.5 0.8],x);
>> subplot(211)
>> stem(n,x,'filled')
>> axis([-5 50 -1 1]); grid
>> ylabel('x[n]')
>> subplot(212)
>> stem(n,y,'filled')
>> axis([-5 50 -30 30]); grid
>> ylabel('y[n]')
>> xlabel('Time Index (n)')
```



- What is the filter gain at $\hat{\omega}_0$?
  - From the plot above we see that in steady-state output peak amplitude over the input peak amplitude is about

$$\text{Gain at } \hat{\omega}_0 \approx \frac{22.66}{1} = 22.66$$

- We can check this by finding the filter frequency response magnitude

```
>> w = [0 0.1*pi]; % input two frequency values
>> H_2pts = abs(freqz(5,[1 -1.5 0.8],w))

H_2pts =  1.6667e+01   2.2652e+01 <= gain at [0,0.1 pi]
```

- The exact gain is 22.65, so the value from the plot is very close

Second-Order Filters

- Second-order IIR filters allow for the possibility of complex conjugate pole and zero pairs, yet still have real coefficients
- The general second-order system function is

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} \tag{8.20}$$

- The corresponding difference equation is

$$\begin{aligned} y[n] = {} & a_1 y[n-1] + a_2 y[n-2] \\ & + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] \end{aligned} \tag{8.21}$$

- The direct-form I and direct-form II structures, discussed on pages 8–2 and 8–13 respectively, can be used to implement (8.21)

Poles and Zeros

- To identify the poles and zeros of $H(z)$ we can first convert to positive powers of z and then factor into poles and zeros

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z + a_2} = b_0 \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

- The coefficients are related to the roots (poles & zeros) via

$$\begin{aligned} b_1 / b_0 &= -(z_1 + z_2) \qquad & b_2 / b_0 &= z_1 z_2 \\ a_1 &= p_1 + p_2 & a_2 &= -p_1 p_2 \end{aligned} \tag{8.22}$$

- Given real coefficients, numerator and denominator, the roots occur either as two real values or as a complex-conjugate pair
- **Poles**: From the quadratic formula

$$p_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

  – Real poles occur when $a_1^2 + 4a_2 > 0$

  – Complex-conjugate poles occur otherwise, and are given by

$$\begin{aligned} p_{1,2} &= \frac{1}{2}a_1 \pm j\frac{1}{2}\sqrt{-a_1^2 - 4a_2} \\ &= re^{\pm j\theta} \end{aligned}$$

  where

$$\begin{aligned} r &= \sqrt{-a_2} \\ \theta &= \cos^{-1}\left(\frac{a_1}{2\sqrt{-a_2}}\right) \end{aligned}$$

- **Zeros**: Similar results hold for the zeros if we factor out b_0 and then replace a_1 with $-b_1/b_0$ and replace a_2 with $-b_2/b_0$

Example: Complex Poles and Zeros

- Consider

$$H(z) = \frac{3 + 2z^{-1} + 2.5z^{-2}}{1 - 1.5z^{-1} + 0.8z^{-2}} = \frac{3\left(1 + \frac{2}{3}z^{-1} + \frac{2.5}{3}z^{-2}\right)}{1 - 1.5z^{-1} + 0.8z^{-2}}$$

- Apply the quadratic formula to the numerator and denominator to find the zeros and poles

$$z_{1,2} = \frac{-2 \pm \sqrt{2^2 - (4 \cdot 3 \cdot 2.5)}}{2 \cdot 3}$$

$$= -0.3333 \pm j0.8498 = 0.9129e^{\pm j1.9446}$$

$$p_{1,2} = \frac{1.5 \pm \sqrt{1.5^2 - (4 \cdot 1 \cdot 0.8)}}{2 \cdot 1}$$

$$= 0.7500 \pm j0.4873 = 0.8944e^{\pm j0.5762}$$

- We can use the MATLAB function `tf2zp()` to convert the system function form to a zero pole form, plus a gain term

```
>> [z,p,K] = tf2zp([3 2 2.5],[1 -1.5 0.8])

z = -3.3333e-01 + 8.4984e-01i % these agree with the
    -3.3333e-01 - 8.4984e-01i % hand calculations
p =  7.5000e-01 + 4.8734e-01i
     7.5000e-01 - 4.8734e-01i
K =  3 % K is the same as b0 in this case
```

Impulse Response

- Using a partial fraction expansion we can inverse transform any rational $H(z)$ back to the impulse response $h[n]$
- Given

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} \quad (8.23)$$

  we first perform long division to reduce the numerator order by one

- It can be shown that the partial fraction expansion will be of the form

$$H(z) = -\frac{b_2}{a_2} + \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} \quad (8.24)$$

  where

$$A_k = H(z)(1 - p_k z^{-1})\Big|_{z = p_k}$$

- The impulse response is found using the table on page 8–24

$$h[n] = -\frac{b_2}{a_2}\delta[n] + A_1 (p_1)^n u[n] + A_2 (p_2)^n u[n] \quad (8.25)$$

- This is a very general result, because the poles may be either real or complex conjugates
  – Note that if $b_2 = 0$ (no long division is required) then the delta function term in (8.25) is not needed

– For complex conjugates poles further simplification is possible because A_1 and A_2 will also be complex conjugates

- **Complex Conjugate Poles**: To simplify (8.25) for this case we first write A_1 and p_1 in polar form

$$\begin{aligned} A_1 &= \alpha e^{j\phi} \\ p_1 &= r e^{j\theta} \end{aligned} \tag{8.26}$$

– We can now write

$$\begin{aligned} A_1(p_1)^n + A_2(p_2)^n &= \alpha r^n e^{j(\theta n + \phi)} + \alpha r^n e^{-j(\theta n + \phi)} \\ &= 2\alpha r^n \cos(\theta n + \phi) \end{aligned} \tag{8.27}$$

and (8.25) specializes to

$$-\frac{b_2}{a_2}\delta[n] + 2\alpha r^n \cos(\theta n + \phi) u[n] \tag{8.28}$$

– We also see from this form that if we place the poles on the unit circle, the impulse response will contain a pure sinusoid, since $r^n \to 1^n = 1$

Example: Conjugate Poles Inside the Unit Circle

- Find the impulse response corresponding to

$$H(z) = \frac{3 + z^{-1}}{\left(1 - \frac{3}{4}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{3}{4}e^{-j\pi/4}z^{-1}\right)}$$

- The partial fraction expansion is of the form

$$H(z) = \frac{A_1}{1 - \frac{3}{4}e^{j\pi/4}z^{-1}} + \frac{A_2}{1 - \frac{3}{4}e^{-j\pi/4}z^{-1}}$$

- We know that $A_2 = A_1^*$

$$A_1 = \left.\frac{3 + z^{-1}}{1 - \frac{3}{4}e^{-j\pi/4}z^{-1}}\right|_{z^{-1} = \frac{4}{3}e^{-j\pi/4}} = \frac{3 + \frac{4}{3}e^{-j\pi/3}}{1 + j}$$

$$= 2.867e^{-j1.020}$$

- Using the general result of (8.28) we see that $\alpha = 2.867$, $\phi = -1.020$, $r = 0.75$, and $\theta = \pi/4$, so

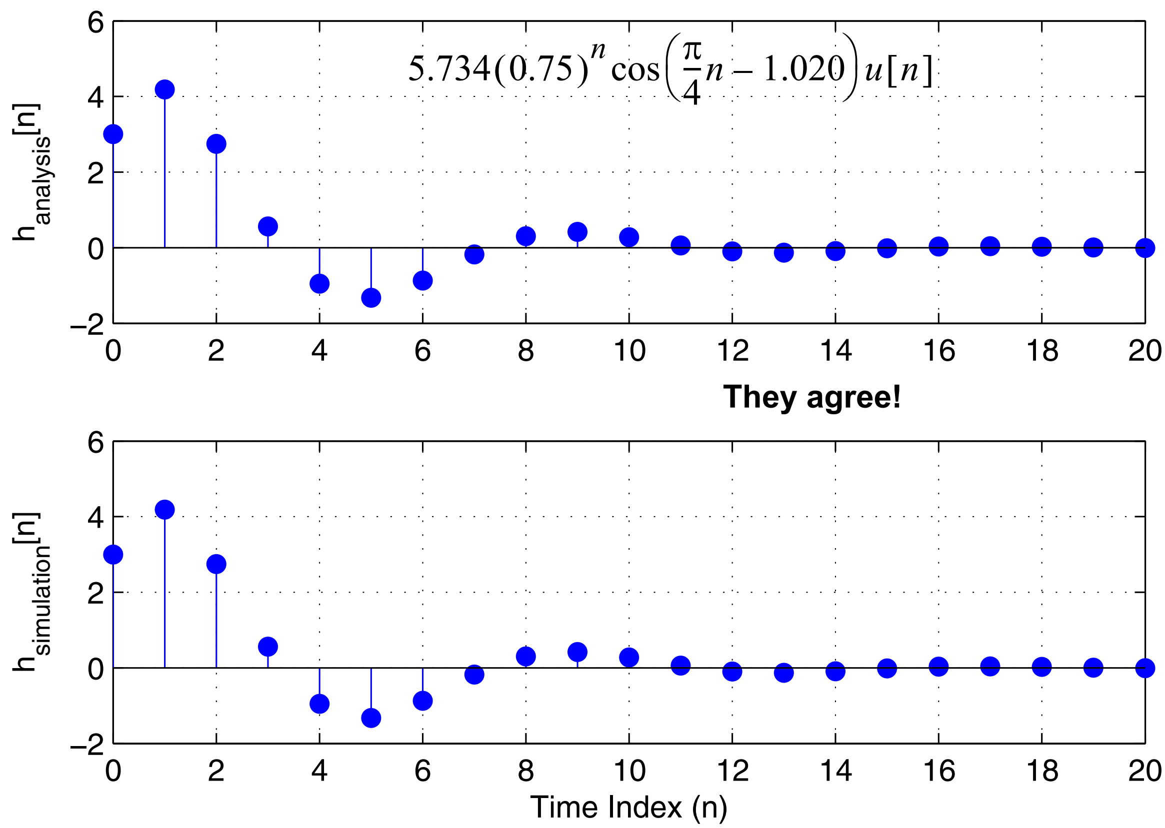
$$h[n] = 2 \cdot 2.867 \cdot (0.75)^n \cos\left(\frac{\pi}{4}n - 1.020\right)$$

- Check using `residuez()`

```
>> [A,p,K] = residuez([3 1],[1 -2*3/4*cos(pi/4) 9/16])

A = 1.5000e+00 - 2.4428e+00i
    1.5000e+00 + 2.4428e+00i

p = 5.3033e-01 + 5.3033e-01i
    5.3033e-01 - 5.3033e-01i

K = []

>> [abs(A(1)) angle(A(1))] % Get mag and angle of A1

ans = 2.8666e+00  -1.0201e+00 % The results agree
```

- Plot $h[n]$ from the above analysis and compare it with a MATLAB simulation using `filter()`

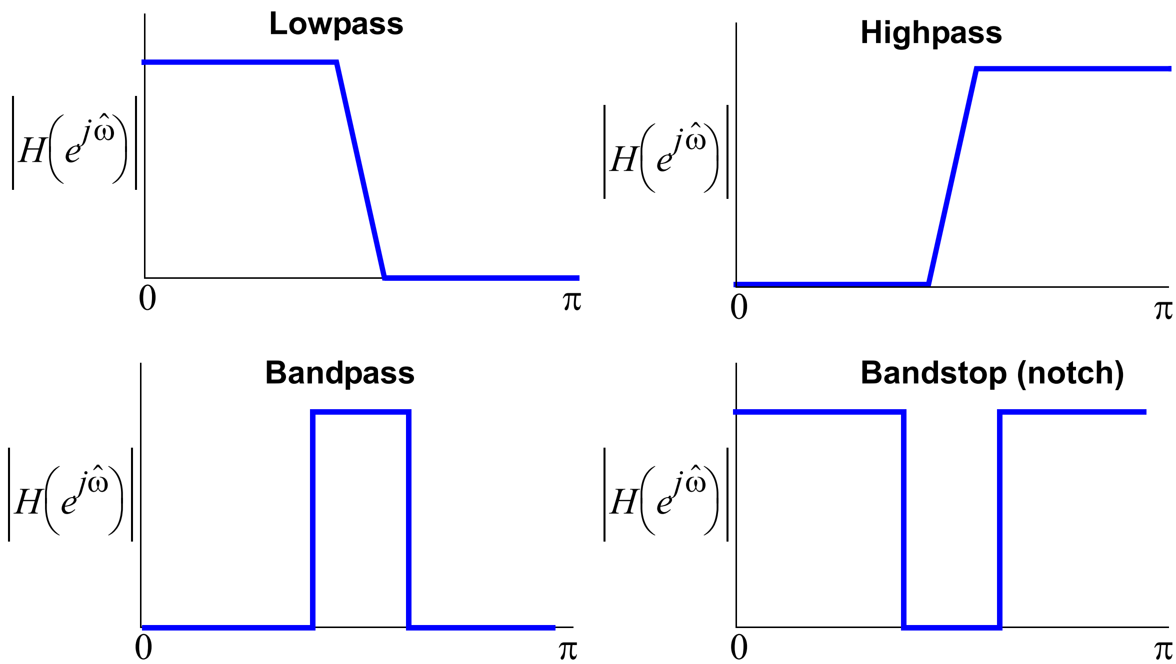
```
>> n = 0:20;
>> h_anal = 2*2.867*(3/4).^n.*cos(pi/4*n - 1.02);
>> subplot(211)
>> stem(n,h_anal,'filled')
>> grid
>> ylabel('h_{analysis}[n]')
>> subplot(212)
>> x = [1 zeros(1,20)];
>> h_sim = filter([3 1], conv([1 -3/4*exp(j*pi/4)],...
                  [1 -3/4*exp(-j*pi/4)]),x);
>> stem(n,h_sim,'filled')
>> grid
>> ylabel('h_{simulation}[n]')
>> xlabel('Time Index (n)')
```



Frequency Response

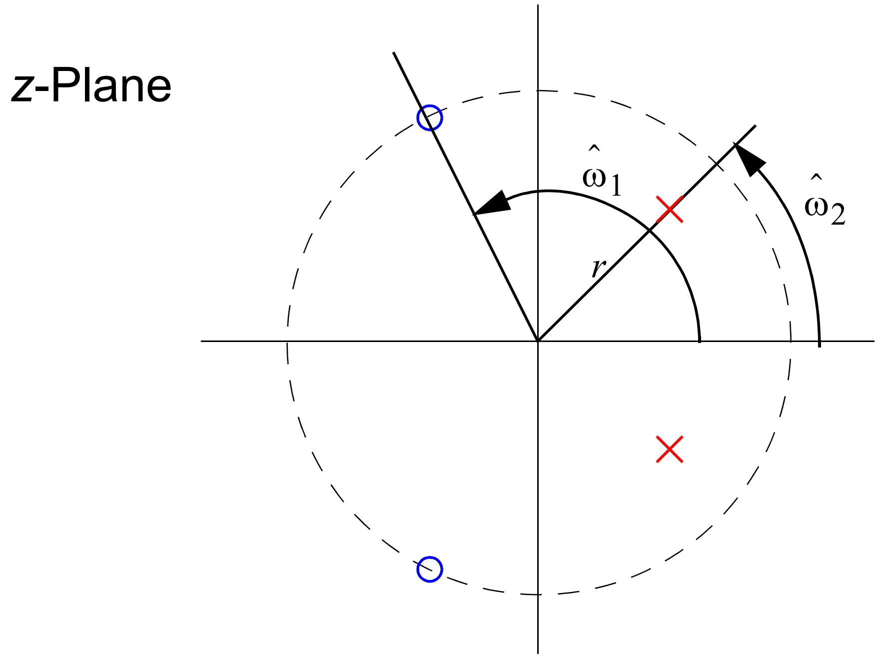
$$H(e^{j\hat{\omega}}) = \frac{b_0 + b_1 e^{-j\hat{\omega}} + b_2 e^{-j2\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}} - a_2 e^{-j2\hat{\omega}}} \tag{8.29}$$

- With just a second-order filter we can realize a frequency response that is lowpass, highpass, bandpass, or bandstop
- Idealized filter responses are shown below



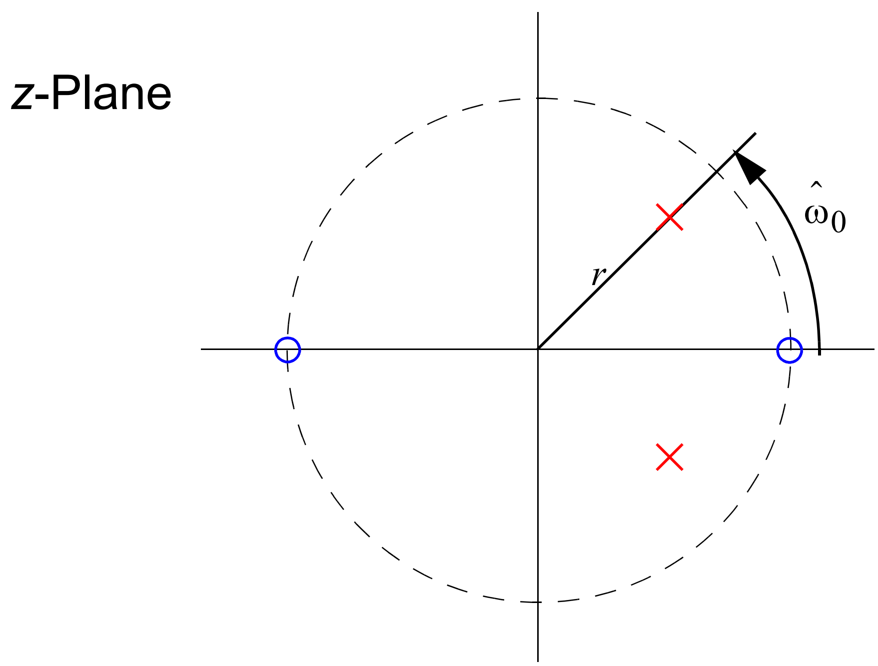
- With just two poles and two zeros the filter action achievable is limited
- **Lowpass/Highpass**: Place zeros on the unit circle as a conjugate pair and poles inside the unit circle as a conjugate pair

$$H(z) = \frac{1 - 2\cos(\hat{\omega}_1)z^{-1} + z^{-2}}{1 - 2r\cos(\hat{\omega}_2)z^{-1} + r^2 z^{-2}} \tag{8.30}$$



- **Bandpass**: Place zeros at $z = 1$ and -1 and poles inside the unit circle as a conjugate pair

$$H(z) = \frac{1 - z^{-2}}{1 - 2r\cos(\hat{\omega}_0)z^{-1} + r^2 z^{-2}} \tag{8.31}$$



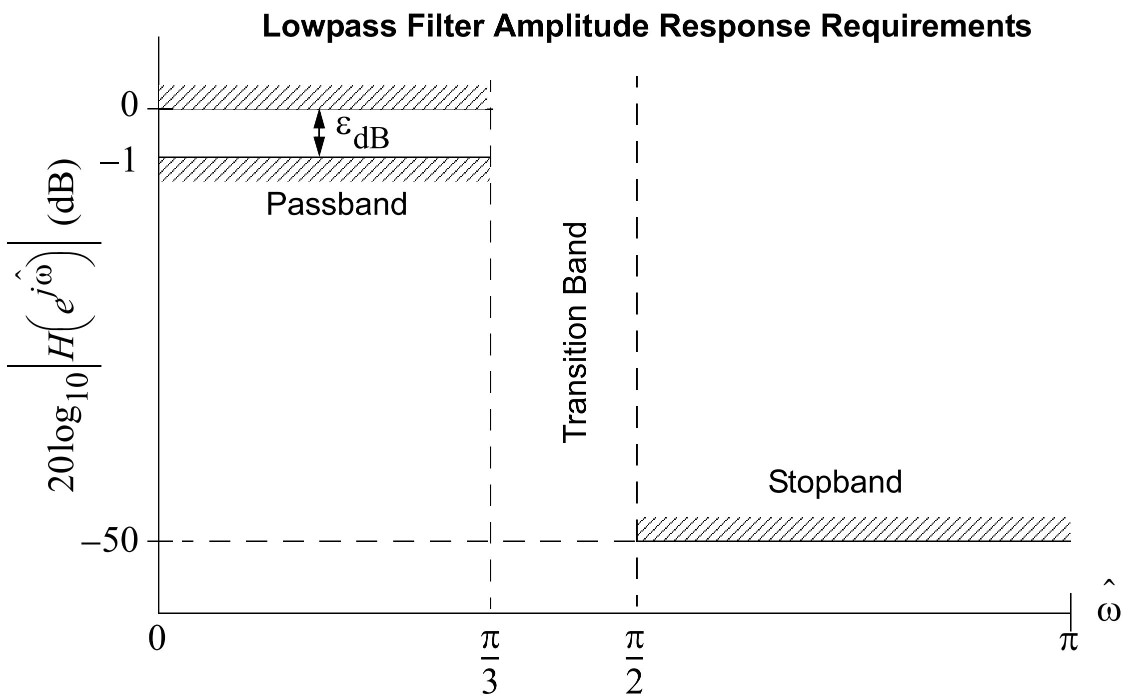
- **Bandstop (notch)**:

$$H(z) = \frac{1 - 2\cos(\hat{\omega}_0)z^{-1} + z^{-2}}{1 - 2r\cos(\hat{\omega}_0)z^{-1} + r^2 z^{-2}} \tag{8.32}$$

  – See the final project

Example of an IIR Lowpass Filter

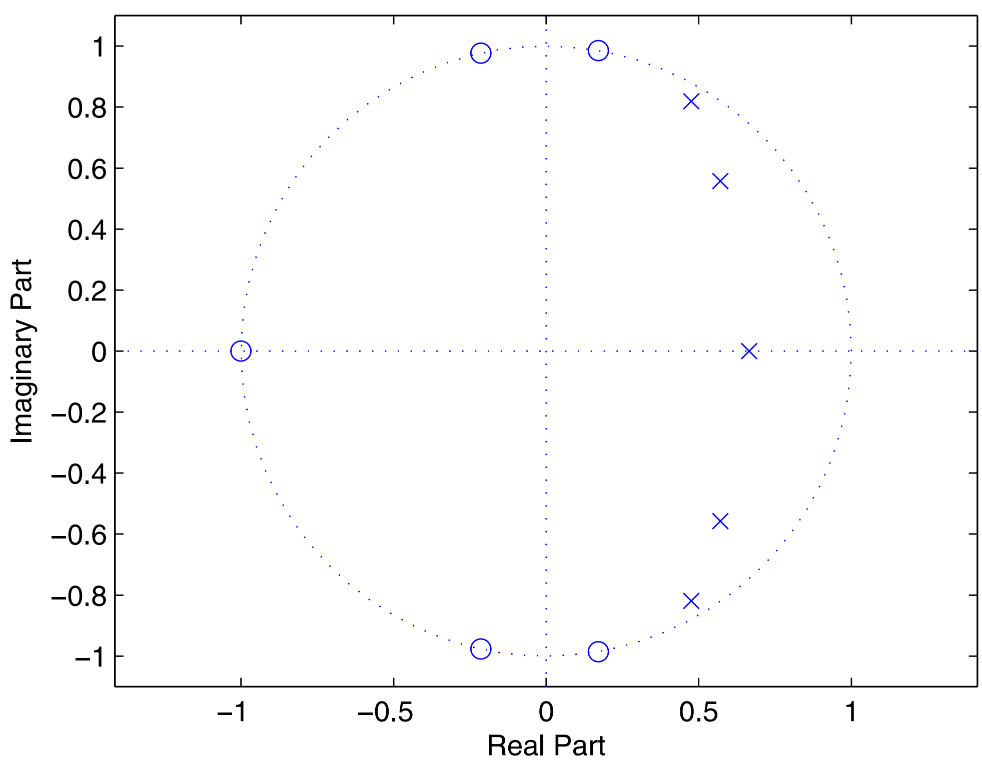
- IIR filters can be designed to meet a desired amplitude response requirement

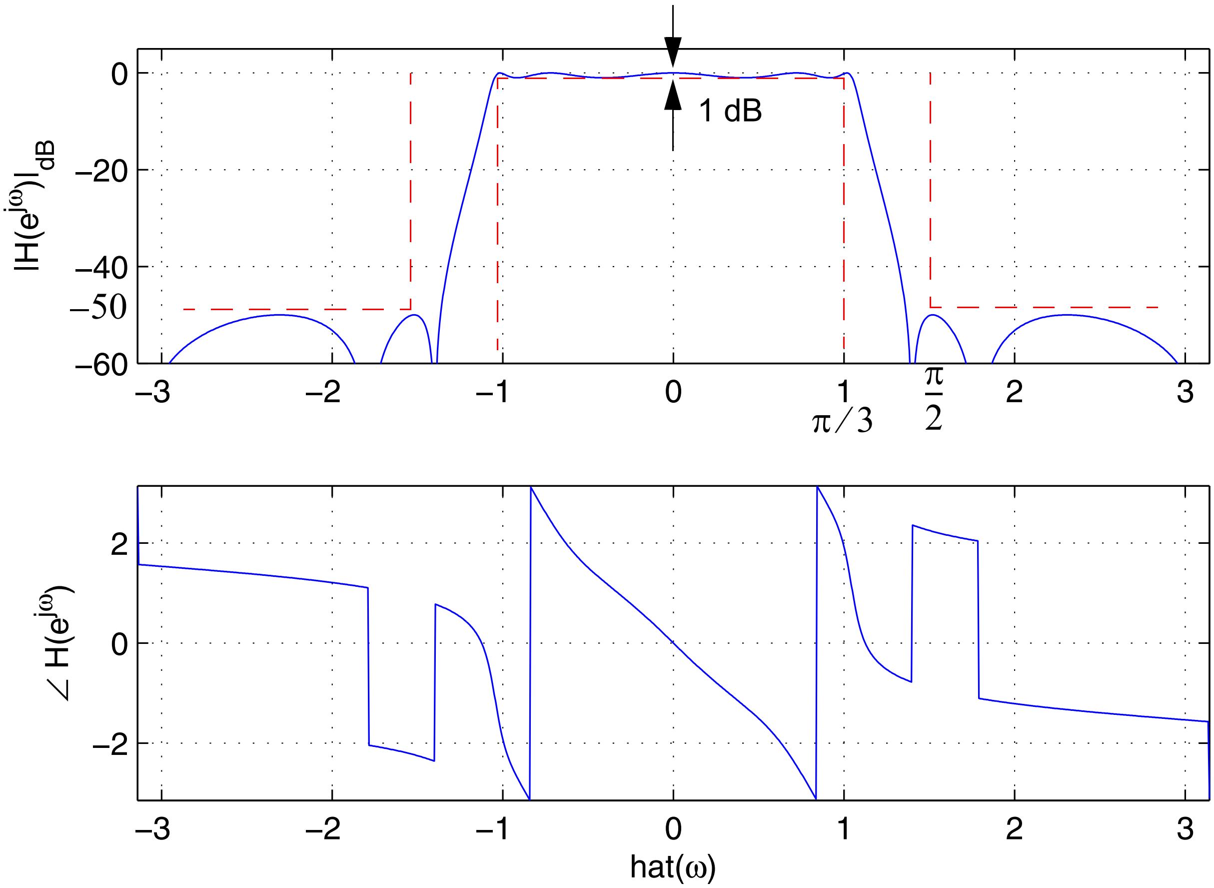


- In the above the filter amplitude response is to remain within a 1 dB tolerance band of 0 dB or unity gain, from $\hat{\omega} = 0$ to $\pi/3$, and have a gain less than -50 dB for $\hat{\omega} > \pi/2$
- This requirement specification defines a lowpass filter

- A variety of IIR filters can be designed to meet these requirements, one such filter type is an elliptic filter, which is available in the MATLAB signal processing toolbox

```
>> ellipord(2*(pi/3/(2*pi)),2*(pi/2/(2*pi)),1,50)

ans = 5 % The required filter order, N, to meet
        % the design requirements.
>> [b,a] = ellip(5,1,50,2*(pi/3/(2*pi)));%design filter
>> b = 1.9431e-02   2.1113e-02   3.7708e-02   3.7708e-02
      2.1113e-02   1.9431e-02 % M=5
>> a   1.0000e+00  -2.7580e+00   4.0110e+00  -3.3711e+00
      1.6542e+00  -3.7959e-01 % N=5
>> zplane(b,a)
>> w = -pi:(pi/500):pi;
>> H = freqz(b,a,w);
>> subplot(211); plot(w,20*log10(abs(H)))
>> subplot(212); plot(w,angle(H))
```





- The fifth-order elliptic design has more than met the amplitude response requirements as it achieves the -50 dB gain level before $\hat{\omega} = \pi/2$
- The complete $H(z)$ is

$$H(z) = \frac{0.01943 + 0.02111z^{-1} + 0.03771z^{-2} + 0.03771z^{-3} + 0.02111z^{-4} + 0.01943z^{-5}}{1 - 2.7580z^{-1} + 4.0110z^{-2} - 3.3711z^{-3} + 1.6542z^{-4} - 0.3796z^{-5}}$$